ALGEBRAIC GEOMETRY 3, WINTER TERM 2023/24. LECTURE COURSE NOTES.

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1. Introduction

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These notes are not complete lecture notes, but should rather be thought of as a rough summary of the content of the course. Many proofs are only sketched, or are omitted. Please do not hesitate to ask for details whenever the given information is not sufficient!

References. The books [GW1], [GW2] by Wedhorn and myself, by Hartshorne [H], and the Stacks project [Stacks]. The book by Vakil [Va] is also recommended. More precise references are given in most of the individual sections.

This lecture course is a continuation of the courses *Algebraic Geometry 1*, *Algebraic Geometry 2* which covered the definition of schemes, some basic notions about schemes and scheme morphisms: Reduced and integral schemes, immersions and subschemes, and fiber products of schemes, separated and proper morphisms, $\mathscr{O}_X$-modules, line bundles and divisors, and basics of the cohomology of $\mathscr{O}_X$-modules including the standard vanishing theorems (Grothendieck vanishing, vanishing of higher cohomology of affine schemes with coefficients in quasi-coherent sheaves) and the finiteness of cohomology for projective schemes.

Outline of this course:

- Smoothness and differentials – The notion of smoothness is very important throughout algebraic geometry, so it is high time that we cover it in the lectures. Furthermore, it is closely related to the notion of differential forms (of course, we need a suitable algebraic form of this). As it will turn out, sheaves of differential forms are in turn closely related to Serre duality, a topic that we have already scratched and that we will come back to in this class.
- Serre duality – We have seen in the last term that Serre duality, while the statement itself is a bit technical, has nice consequences such as the Theorem of Riemann-Roch. With the theory of differentials at hand, it will not be so difficult to develop the cohomological machinery until the point where we can prove it for a large class of schemes.
- Cohomology and base change – Another crucial technique to study cohomology goes under the name of cohomology and base changes.

It concerns the following question: Given a morphism $f\colon X \to Y$ of schemes, and a quasi-coherent $\mathscr{O}_X$-modules $\mathscr{F}$, under which conditions is the natural $\kappa(y)$-vector space homomorphism

$$R^i f_* \mathscr{F} \otimes \kappa(y) \to H^i(X_y, \mathscr{F}_{|X_y})$$

an isomorphism? Here $y \in Y$ and $X_y := X \times_Y \operatorname{Spec} \kappa(y)$ is the scheme-theoretic fiber of f over y.

- Grassmannians, flag varieties, Schubert varieties – These are classes of varieties that have a rather explicit definition, and on the other hand are quite interesting and occur in many different contexts. Grassmannians are natural generalizations of projective space; they parameterize r-dimensional subspaces of an n-dimensional vector space. (I.e., for $r = 1$ we obtain the projective space $\mathbb{P}^{n-1}$.)
- Hilbert schemes – If there is time, I will discuss the construction of the Hilbert scheme. Similarly as for projective space and for Grassmannians, it is (relatively) easy to write down the functor of T-valued points of the Hilbert scheme. However, in this case it is far from obvious that a scheme giving rise to this functor exists. A crucial ingredient of the proof is the cohomologocal machinery we have developed, in particular the theory of cohomology and base change.

I want to give some pointers to results that answer natural questions that can be answered by the above tools (in particular, the machinery of cohomology), and which can be stated without any reference to this.

Remark: Quasi-projectivity of curves 1.1. Let k be a field. In this remark, by a *curve* over k we mean a separated finite type k-scheme of dimension 1, i.e., every local ring of a closed point has dimension 1.

Theorem. Let C be a curve. Then C is a quasi-projective k-scheme, i.e., C is isomorphic to a locally closed subscheme of some projective space.

We will not discuss the complete proof of the theorem here, but only sketch some of the steps.

(I) Assume that C is normal, i.e., all local rings of C at closed points are discrete valuation rings. The key point is then that the projective space $\mathbb{P}^n_k$ satisfies the *valuative criterion of properness*:

Theorem 1.2. (Valuative criterion of properness, noetherian version, [GW1] Theorem 15.9)

Let S be a noetherian scheme and let $f\colon X \to Y$ be a morphism of finite type. We consider commutative diagrams of the form

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y, \end{array}$$

where R is a discrete valuation ring with field of fractions K and the vertical arrow on the left is the canonical inclusion.

The following are equivalent:

(i) *The morphism f is proper (resp., separated).*

(ii) *In every diagram as above there exists a unique (resp., at most one) morphism* $\operatorname{Spec} R \to X$ *making the resulting diagram commutative.*

We have proved in AG2 that the structure morphism $\mathbb{P}^n_k \to \operatorname{Spec} k$ is proper. It follows that it has the property stated in the criterion. However, since we did not prove the valuative criterion of properness, in class we verified directly that it holds for $\mathbb{P}^n_k$ over k. This is easy using the description of S-valued points of projective space; for S the spectrum of a local ring, one basically obtains a description in terms of homogeneous coordinates (cf. [GW1] Exer. 4.6 for a precise statement). Proving (ii) $\Rightarrow$ (ii) in the valuative criterion would now give a different proof of the fact that projective space is proper over the base.

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Now let $U \subseteq C$ be open affine and choose an immersion $f\colon U \hookrightarrow \mathbb{A}^n_k \hookrightarrow \mathbb{P}^n_k$. For $x \in C \setminus U$, the above shows that we can extend the morphism $\operatorname{Spec} K(C) \to \mathbb{P}^n_k$ given by f to a morphism $\operatorname{Spec} \mathscr{O}_{C,x} \to \mathbb{P}^n_k$. This morphism can be extended to some open neighborhood V of x (view $\mathscr{O}_{C,x}$ as the localization of $\Gamma(V, \mathscr{O}_C)$ with respect to some prime ideal; the image of $\operatorname{Spec} \mathscr{O}_{C,x}$ is contained in one of the standard open charts of $\mathbb{P}^n_k$, which we can write as $k[X_1, \dots, X_n]$; in the images of the X_i in $\mathscr{O}_{C,x}$ only finitely many denominators are involved, hence on the ring level the homomorphism factors through the localization with respect to a suitable element s; for the scheme morphism this means that it extends to $D(s) \subseteq V$; see [GW1] Prop. 10.52). Since C is reduced and $\mathbb{P}^n_k$ is separated, and the morphisms $U \to \mathbb{P}^n_k$ and $V \to \mathbb{P}^n_k$ coincide on $\operatorname{Spec} K(C)$, which is dense, they actually coincide on $U \cap V$ (AG2, Problem [GW1] Cor. 9.9) and we can glue them. Repeating this, if necessary, we can extend the morphism $U \to \mathbb{P}^n_k$ to a morphism $C \to \mathbb{P}^n_k$.

At this point, the choice of an affine open U and an immersion $U \to \mathbb{P}^n_k$ gives us a morphism $C \to \mathbb{P}^n_k$. However, in general this will not be an immersion. To finish the proof of the theorem for normal curves, we proceed as follows. Let $C = \bigcup_{i=1}^m U - i$ be an affine open cover. For each U_i, as above we obtain a morphism $f_i\colon C \to \mathbb{P}^{n_i}_k$ such that $f_{i|U_i}$ is an immersion. This gives us a morphism $C \to \prod_i \mathbb{P}^{n_i}_k$ 8where the product is the fiber product over $\operatorname{Spec} k$), and composing this morphism with the Segre embedding $\prod_i \mathbb{P}^{n_i}_k \to \mathbb{P}^N_k$ (where N depends on the n_i as dictated by the Segre embedding, a closed embedding; see [GW1] Section (4.14)) we obtain a morphism $f\colon C \to \mathbb{P}^N_k$ such that for every i, the restriction $f_{|U_i}$ is an immersion. We can then conclude by the following lemma.

Lemma 1.3. ([GW1] Lemma 14.18) *Let X be a scheme which has only finitely many irreducible components. Let $f\colon X \to Y$ be a separated morphism. Assume that there exists a cover $X = \bigcup_i U_i$ where each U_i is open* *and dense in X, and such that for every i the restriction $f_{|U_i}$ is an (open) immersion. Then f is an (open) immersion.*

Proof. The issue here is to show that under the given assumptions, f is injective.

We sketch the proof in case X is irreducible and all $f_{|U_i}$ are open immersions, which is the case relevant for us. By replacing Y with the reduced closure of the image of f, we reduce to the case that Y is integral and that f is dominant (i.e., the image of f is dense in Y). It follows that for every $x \in X$, the ring homomorphism $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is an isomorphism and thus in particular *flat*.

That all these ring homomorphisms are flat is usually expressed by saying that f is flat. This property is stable under base change. We will use that whenever $\varphi\colon A \to B$ is a flat local ring homomorphism of local rings, then the induced morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective; this is a commutative algebra result which sometimes goes by the name *going down for flat morphisms*, see e.g. [GW1] Example B.18, [M2] Theorem 9.5. A *maximal point* of a scheme is a point x such that there exists no point $x' \neq x$ with $x \in \overline{\{x'\}}$, i.e., x is a generic point of an irreducible component. As a consequence of the above discussion we see that under a flat morphism $X \to Y$ of schemes, every maximal point of X is mapped to a maximal point of Y:

Coming back to the specific situation at hand, to show that f is injective, we will show that the diagonal morphism $\Delta\colon X \to X \times_Y X$ is surjective. The injectivity of f is an easy consequence of this. Since by assumption f is separated, Δ is a closed immersion, thus it is enough to show that all maximal points of $X \times_Y X$ are in its image.

So let $\zeta \in X \times_Y X$ be maximal. Since the projections $X \times_Y X \to X$ and f are flat morphisms, ζ maps to the unique maximal point of X under the projections, and to the unique maximal point in Y under the composition with f. Looking at local rings, we obtain a commutative square

$$\begin{array}{ccc} \mathscr{O}_{X\times_Y X,\zeta} & \longleftarrow & K(X) \\ \uparrow & & \uparrow \\ K(X) & \longleftarrow & K(Y). \end{array}$$

But since the maximal point of X lies in each of the subsets U_i, our assumptions imply that $K(X) = K(Y)$. This implies that the morphism $\operatorname{Spec} \mathscr{O}_{X\times_Y X,\zeta} \to X\times_Y X$ factors through $\operatorname{Spec} K(X)\otimes_{K(Y)} K(X) = \operatorname{Spec} K(X)$ and hence through Δ, which shows that ζ is in the image of Δ. $\square$

(II) With a bit more work, this strategy can be extended to cover all reduced curves, see [GW1] Theorem 15.18.

(III) For the general case, we use the description of S-valued points of projective space in terms of line bundles. We search for a line bundle $\mathscr{M}$

on the given curve C together with a surjection $\mathscr{O}_C^{n+1} \to \mathscr{L}$ such that the corresponding morphism $C \to \mathbb{P}^n_k$ is an immersion.

Let C_{red} be the underlying reduced subscheme of C and by $\iota\colon C_{\text{red}} \to C$ the corresponding closed immersion. By step (II) we know, that on C_{red} a line bundle $\mathscr{L}'$ (together with a surjection $\mathscr{O}_{C_{\text{red}}}^{n'} \to \mathscr{L}'$) with the desired property exists. The crucial step then, is to show that there exists a line bundle $\mathscr{L}$ on C with $\iota^*\mathscr{L} \cong \mathscr{L}'$. One can then show that from such an $\mathscr{L}$ one can construct $\mathscr{M}$ as desired (in fact, this is also an application of cohomological methods, namely "Serre's criterion for ampleness"); we skip this step here.

To proceed, we need the following cohomological description of the Picard group of a scheme.

Proposition 1.4. *Let X be a scheme.*

(1) *Let $\mathscr{U} = (U_i)_i$ be an open cover of X. Then the Čech cohomology group $H^1(\mathscr{U}, \mathscr{O}_X^\times)$ can be identified with the subgroup of* $\operatorname{Pic}(X)$ *consisting of isomorphism classes of line bundles $\mathscr{L}$ such that $\mathscr{L}_{|U_i} \cong \mathscr{O}_{U_i}$ for all i.*
(2) *We have an isomorphism* $\operatorname{Pic}(X) \cong H^1(X, \mathscr{O}_X^\times)$.

The existence of $\mathscr{L}$ follows from the following lemmas together with the Grothendieck vanishing theorem.

Lemma 1.5. *Let X be a scheme. Then there is a natural isomorphism* $\operatorname{Pic}(X) \cong H^1(X, \mathscr{O}_X^\times)$.

Proof. We can compute the H^1 as Čech cohomology. The key point is then the observation that we can identify, for an open cover $\mathscr{U} = (U_i)_{i\in I}$ of X, the Čech cohomology group $H^1(\mathscr{U}, \mathscr{O}_X^\times)$ and the subgroup of the Picard group given by isomorphism classes of line bundles $\mathscr{L}$ on X such that $\mathscr{L}_{|U_i} \cong \mathscr{O}_{U_i}$ for every i. Cf. [GW1] Sections (11.5), (11.7). □

Lemma 1.6. *Let $i\colon X_0 \to X$ be a closed immersion of schemes defined by a quasi-coherent ideal $\mathscr{I} \subset \mathscr{O}_X$ with $\mathscr{I}^2 = 0$ so that we can view $\mathscr{I}$ as $\mathscr{O}_{X_0}$-module. Then there exists an exact sequence of abelian groups*

$$H^1(X_0, \mathscr{I}) \to H^1(X, \mathscr{O}_X^\times) \to H^1(X_0, \mathscr{O}_{X_0}^\times) \to H^2(X_0, \mathscr{I}). \tag{1.0.1}$$

Proof. See Problem sheet 1. □

See [GW2] Theorem 26.16 for references to the missing pieces.

Remark: Projectivity of surfaces 1.7. Let k be a field, and let X be a separated k-scheme of finite type of dimension 2 (i.e., all local rings at closed points have dimension 2). We call X a *surface*.

Theorem. If X is regular (i.e., all local rings of X are regular local rings), then X can be embedded into some projective space over k as a locally closed subscheme.

To prove the theorem, by “Nagata’s compactification theorem” (itself a difficult theorem, see [GW1] Section (12.15) for its statement and references) one can restrict to the case that X is a proper k-scheme.

For X proper, a key ingredient is the Lemma of Enriques-Severi-Zariski that we have seen at the end of the Algebraic Geometry 2 class. See [GW2] Theorem 25.151 for a proof of the theorem (which requires quite a few other ingredients, many of them also relying on cohomology).

The regularity assumption cannot be dropped in the theorem. In higher dimension, the corresponding statement fails even for regular k-schemes.

Another example of a topic (among very many others ...) that illustrate the use of “heavy machinery” (in particular, cohomological methods) are the *Weil conjecturs for curves over finite fields*, see [GW2] Sections (26.28), (26.29).

2. Smoothness and differentials

Oct. 16, 2023

General reference: [GW1] Ch. 6.

The Zariski tangent space.

(2.1) Definition of the Zariski tangent space.

Definition 2.1. *Let X be a scheme, $x \in X$, $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ the maximal ideal in the local ring at x, $\kappa(x)$ the residue class field of X in x. The $\kappa(x)$-vector space $(\mathfrak{m}/\mathfrak{m}^2)^*$ is called the* (Zariski) tangent space of X in x.

Definition 2.2. *Let R be a ring, $f_1, \dots, f_r \in R[T_1, \dots, T_n]$. We call the matrix*

$$J_{f_1,\dots,f_r} := \left(\frac{\partial f_i}{\partial T_j}\right)_{i,j} \in M_{r\times n}(R[T_\bullet])$$

the Jacobian matrix *of the polynomials f_i. Here the partial derivatives are to be understood in a formal sense.*

Remark 2.3.
(1) If in the above setting the ideal $\mathfrak{m}$ is finitely generated, then $\dim_{\kappa(x)} T_xX$ is the minimal number of elements needed to generate $\mathfrak{m}$ and in particular is finite.
(2) The tangent space construction is functorial in the following sense: Given a scheme morphism $f\colon X \to Y$ and $x \in X$ such that $\dim_{\kappa(f(x))} T_{f(x)}Y$ is finite or $[\kappa(x) : \kappa(f(x))]$ is finite, then we obtain a map
$$df_x\colon T_xX \to T_{f(x)}Y \otimes_{\kappa(f(x))} \kappa(x).$$

Example 2.4. Let k be a field, $X = V(f_1, \dots, f_m) \subseteq \mathbb{A}^n_k$, $f_i \in k[T_1, \dots, T_n]$, $x = (x_i)_i \in k^n = \mathbb{A}^n(k)$. Then there is a natural identification $T_xX = \operatorname{Ker}(J_{f_1,\dots,f_m}(x))$, where $J_{f_1,\dots,f_m}(x)$ denotes the matrix with entries in $\kappa(x) = k$ obtained by mapping each entry of $J_{f_1,\dots,f_m}$ to $\kappa(x)$, which amounts to evaluating these polynomials at x.

Proposition 2.5. *Let k be a field, X a k-scheme, $x \in X(k)$. There is an identification (functorial in (X, x))*

$$X(k[\varepsilon]/(\varepsilon^2))_x := \{f \in \operatorname{Hom}_k(\operatorname{Spec} k[\varepsilon]/(\varepsilon^2), X);\ \operatorname{Im}(f) = \{x\}\} = T_xX.$$

Remark 2.6.
(1) It is possible to define the k-vector space structure on $X(k[\varepsilon]/(\varepsilon^2))_x$ "directly".
(2) Similarly, one can define the relative tangent space of an S-scheme X in a K-valued point ξ for any field K and without restrictions on the residue class field of the image point of ξ, as the set of S-morphisms $f\colon \operatorname{Spec} K[\varepsilon]/(\varepsilon^2) \to X$ with $\operatorname{Im}(f) = \operatorname{Im}(\xi)$ (and again, this set can be

made into a K-vector space). This concept is sometimes useful, but the result is in general different from the Zariski tangent space.

Smooth morphisms.

(2.2) Dimension of schemes.

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Reference: [GW1] Sections (5.3) ff.

Recall from commutative algebra that for a ring R we define the (Krull) dimension $\dim R$ of R as the supremum over all lengths of chains of prime ideals, or equivalently as the dimension of the topological space $\operatorname{Spec} R$ in the sense of the following definition.

Definition 2.7. *Let X be a topological space. We define the dimension of X as*

$$\begin{aligned}\dim X := \sup\{\ell;\ & \textit{there exists a chain } Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_\ell \\ & \textit{of closed irreducible subsets } Z_i \subseteq X\}.\end{aligned}$$

We will use this notion of dimension for non-affine schemes, as well. Recall the following theorem about the dimension of finitely generated algebras over a field from commutative algebra:

Theorem 2.8. *Let k be a field, and let A be a finitely generated k-algebra which is a domain. Let $\mathfrak{m} \subset A$ be a maximal ideal. Then*

$$\dim A = \operatorname{trdeg}_k(\operatorname{Frac}(A)) = \dim A_{\mathfrak{m}}.$$

By passing to an affine cover, we obtain the following corollary:

Corollary 2.9. *Let k be a field, and let X be an integral k-scheme which is of finite type over k. Denote by $K(X)$ its field of rational functions. Let $U \subseteq X$ be a non-empty open subset, and let $x \in X$ be a closed point. Then*

$$\dim X = \dim U = \operatorname{trdeg}_k(K(X)) = \dim \mathscr{O}_{X,x}.$$

(2.3) Definition of smooth morphisms.

Reference: [GW1] Section (6.8).

Definition 2.10. *A morphism $f\colon X \to Y$ of schemes is called* smooth of relative dimension $d \geq 0$ in $x \in X$, *if there exist affine open neighborhoods $U \subseteq X$ of x and $V = \operatorname{Spec} R \subseteq Y$ of $f(x)$ such that $f(U) \subseteq V$ and an open immersion $j\colon U \to \operatorname{Spec} R[T_1, \dots, T_n]/(f_1, \dots, f_{n-d})$ such that the triangle*

$$\begin{tikzcd} U \arrow[rr, "j"] \arrow[dr, "f"'] & & \operatorname{Spec} R[T_1,\dots,T_n]/(f_1,\dots,f_{n-d}) \arrow[dl] \\ & V & \end{tikzcd}$$

is commutative, and that the Jacobian matrix $J_{f_1,\dots,f_{n-d}}(x)$ *has rank* $n-d$.

We say that $f\colon X \to Y$ *is* smooth of relative dimension d *if* f *is smooth of relative dimension* d *at every point of* X. *Instead of* smooth of relative dimension 0, *we also use the term* étale.

With notation as above, if f is smooth at $x \in X$, then x has an open neighborhood such that f is smooth at all points of this open neighborhood. Clearly, $\mathbb{A}^n_S$ and $\mathbb{P}^n_S$ are smooth of relative dimension n for every scheme S. (It is harder to give examples of non-smooth schemes directly from the definition; we will come back to this later.)

Remark 2.11. (The Jacobian Conjecture) Let k be a field, $n \geq 1$, and let $f_1, \dots, f_n \in k[X_1, \dots, X_n]$. The f_i define a k-scheme morphism $\mathbb{A}^n_k \to \mathbb{A}^n_k$, given on R-valued points by $(x_1, \dots, x_n) \mapsto (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots x_n))$.

Assume that f is an isomorphism of k-schemes. It then follows easily, by similar computations as above (or expressed differently by the "multi-variable chain rule"), that the Jacobian matrix of the f_i is invertible in $\mathrm{Mat}_{n\times n}(k[X_\bullet])$. Equivalently, the determinant of the Jacobian matrix lies in $k^\times$.

Jacobian conjecture (O. Keller, 1939) Let k be a field of characteristic 0, $n \geq 1$, and let $f_1, \dots, f_n \in k[X_1, \dots, X_n]$. The morphism $\mathbb{A}^n_k \to \mathbb{A}^n_k$ induced by the f_i is an isomorphism if and only if the Jacobian matrix $\left(\frac{\partial f_i}{\partial X_i}\right)_{i,j} \in \mathrm{Mat}_{n\times n}(k[X_\bullet])$ is invertible.

For $n = 1$ the statement is easy to prove, but the conjecture is open even for $n = 2$ and is particularly well-known for the number of incorrect attempts of proving it.

It is not very hard to see that the condition that k has characteristic 0 cannot be dropped. Can you find an example for this?

With a bit of effort, one can show that equivalently, one can formulate the conjecture as follows: Let k be a field of characteristic 0, and let $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ be an étale morphism. Then f is an isomorphism.

(2.4) Existence of smooth points.

Reference: [GW1] Section (6.9).

Let k be a field.

Lemma 2.12. ([GW1] Lemma 6.17, Prop. 10.52) *Let* X, Y *be [integral*[1]*] k-schemes which are locally of finite type over k. Let* $x \in X$, $y \in Y$, *and let* $\varphi\colon \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ *be an isomorphism of k-algebras. Then there exist open neighborhoods* U *of* x *and* V *of* y *and an isomorphism* $h\colon U \to V$ *of k-schemes with* $h^\sharp_x = \varphi$.

[1] The statement is true in general, but in the lecture we proved it only with the additional assumption that X and Y are integral.

Proposition 2.13. *Let X be an integral k-scheme of finite type. Assume that $K(X) \cong k(T_1, \dots, T_d)[\alpha]$ with α separable algebraic over $k(T_1, \dots, T_d)$. (This is always the case, if k is perfect.) (Then $\dim X = d$ by the above.)*

Then there exists a dense open subset $U \subseteq X$ and a separable irreducible polynomial $g \in k(T_1, \dots, T_d)[T]$ with coefficients in $k[T_1, \dots, T_d]$, such that U is isomorphic to a dense open subset of $\operatorname{Spec} k[T_1, \dots T_d]/(g)$.

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Theorem 2.14. *Let k be a perfect field, and let X be a nonempty reduced k-scheme locally of finite type over k. Then the smooth locus*

$$X_{\mathrm{sm}} := \{x \in X;\ X \to \operatorname{Spec} k \text{ is smooth at } x\}$$

of X is open and dense.

(2.5) Regular rings.

Definition 2.15. *A noetherian local ring A with maximal ideal $\mathfrak{m}$ and residue class field κ is called* regular*, if $\dim A = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.*

One can show that the inequality $\dim A \le \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ always holds. Therefore we can rephrase the definition as saying that A is regular if $\mathfrak{m}$ has a generating system consisting of $\dim A$ elements.

Definition 2.16. *A noetherian ring A is called* regular*, if $A_\mathfrak{m}$ is regular for every maximal ideal $\mathfrak{m} \subset A$.*

We quote the following (mostly non-trivial) results about regular rings. A key input for Part (4) is a version of Krull's Principal Ideal Theorem.

Theorem 2.17. (See [GW1] Proposition B.77 for precise references, [M2], [AM] Ch. 11)

(1) *Every localization of a regular ring is regular.*
(2) *If A is regular, then the polynomial ring $A[T]$ is regular.*
(3) *(Theorem of Auslander–Buchsbaum) Every regular local ring is a unique factorization domain.*
(4) *Let A be a regular local ring with maximal ideal $\mathfrak{m}$ and of dimension d, and let $f_1, \dots, f_r \in \mathfrak{m}$. Then $A/(f_1, \dots, f_r)$ is regular of dimension $d - r$ if and only if the images of the f_i in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $A/\mathfrak{m}$.*

Note that Part (3) implies in particular that every regular local ring is a domain. The UFD property also implies that this domain is integrally closed in its field of fractions.

(2.6) Smoothness and regularity.

Reference: [GW1] Section (6.12).

Let k be a field.

Lemma 2.18. *Let X be a k-scheme locally of finite type. Let $x \in X$ such that $X \to \operatorname{Spec} k$ is smooth of relative dimension d in x. Then $\mathscr{O}_{X,x}$ is regular of dimension $\le d$. If moreover x is closed, then $\mathscr{O}_{X,x}$ is regular of dimension d.*

Sketch of proof. First, reduce to the case that (1) x is a closed point in X. By the definition of smooth morphisms, it is then enough to consider the case of a closed point $x \in \operatorname{Spec} k[X_\bullet](f_\bullet)$ where the Jacobian matrix has full rank. By Theorem 2.17 (2) and (4) it is enough to show that the images of the f_i in $\mathfrak{m}_x/\mathfrak{m}_x^2$ are linearly independent. This is clear (cf. Example 2.4) if $\kappa(x) = k$, and the general case can be reduced to this one, using that in the base change $X \otimes_k \kappa(x)$ there exists a point $\overline{x}$ with residue class field $\kappa(x)$ projecting to $x \in X$ and that we have an inclusion $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathfrak{m}_{\overline{x}}/\mathfrak{m}_{\overline{x}}^2$ of $\kappa(x)$-vector spaces. □

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Lemma 2.19. *Let $X = V(g_1, \dots, g_s) \subseteq \mathbb{A}^n_k$, and let $x \in X$ be a closed point. If $\operatorname{rk} J_{g_1,\dots,g_s}(x) \ge n - \dim \mathscr{O}_{X,x}$, then x is smooth at X/k, and $\operatorname{rk} J_{g_1,\dots,g_s}(x) = n - \dim \mathscr{O}_{X,x}$.*

Sketch of proof. Write $d = \dim \mathscr{O}_{X,x}$. After renumbering the g_i, if necessary, we may assume that the first $n-d$ columns of $J_{g_\bullet}(x)$ are linearly independent. We then have

$$x \in X = V(g_1, \dots, g_s) \subseteq Y := V(g_1, \dots, g_{n-d}) \subseteq \mathbb{A}^n_k,$$

and x is smooth over k as a point of Y. By the previous lemma, $\dim \mathscr{O}_{Y,x} = d$. It follows that $\mathscr{O}_{X,x} = \mathscr{O}_{Y,x}$, and together with Lemma 2.12 we obtain the claim. □

Lemma 2.20. ([GW1] Corollary 5.47) *Let X be a k-scheme locally of finite type and let $x \in X$ be a closed point. Fix an algebraically closed extension field K of k and write $X_K = X \otimes_k K$. If $\overline{x} \in X_K$ is a point mapping to x, then*

$$\dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{X_K,\overline{x}}.$$

Very sketchy indications of proof. For $\ge$ choose some affine open neighborhood of x, apply Noether normalization, and use that the properties *finite* and *injective* of a ring homomorphism are preserved under the base change $- \otimes_k K$.

For $\le$, use that the morphism $X_K \to X$ (being obtained by base change from $\operatorname{Spec}(K) \to \operatorname{Spec}(k)$) is flat, and that flat ring homomorphisms satisfy a *going down* theorem. The key fact for the going down property is that for every flat local ring homomorphism $A \to B$ between local rings, the map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective. Cf. [GW1] Lemma 14.9 or [M2] Theorem 7.3, Theorem 9.5. (In [GW1], the proof of $\le$ is given using the more difficult Proposition 5.44/Theorem 14.38 there, which is required in the book anyway; but at this point the above, related but simpler method works.) □

Theorem 2.21. *Let X be a k-scheme locally of finite type, $x \in X$ a closed point, $d \geq 0$. Fix an algebraically closed extension field K of k and write $X_K = X \otimes_k K$. The following are equivalent:*
(i) *The morphism $X \to \operatorname{Spec} k$ is smooth of relative dimension d at x.*
(ii) *For all points $\overline{x} \in X_K$ lying over x, X_K is smooth over K of relative dimension d at $\overline{x}$.*
(iii) *There exists a point $\overline{x} \in X_K$ lying over x, such that X_K is smooth over K of relative dimension d at $\overline{x}$.*
(iv) *For all points $\overline{x} \in X_K$ lying over x, the local ring $\mathscr{O}_{X_K,\overline{x}}$ is regular of dimension d.*
(v) *There exists a point $\overline{x} \in X_K$ lying over x, such that the local ring $\mathscr{O}_{X_K,\overline{x}}$ is regular of dimension d.*

If these conditions hold, then the local ring $\mathscr{O}_{X,x}$ is regular of dimension d, and if $\kappa(x) = k$, then this last condition is equivalent to the previous ones.

Sketch of proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are easy.

Furthermore (iii) $\Rightarrow$ (iv) and the regularity of $\mathscr{O}_{X,x}$ for a smooth point x follow from Lemma 2.18.

Next we show that the regularity of $\mathscr{O}_{X,x}$ implies that x is a smooth point if $\kappa(x) = k$. Write $d = \dim \mathscr{O}_{X,x} = \dim_k T_x X$. We embed an affine open neighborhood U into affine space, say as an open subscheme of $V(g_1, \dots, g_s) \subseteq \mathbb{A}^n_k$. We are then in the situation of Lemma 2.19, and the lemma shows that x is a smooth point. This also shows (v) $\Rightarrow$ (iii).

It remains to prove that (iii) $\Rightarrow$ (i). It is enough to consider the case where x is a closed point of $V(g_1, \dots, g_s) \subset \mathbb{A}^n_k$ for some polynomials g_i. By Lemma 2.19, it is enough to show that $\operatorname{rk} J_{g_\bullet}(x) = n - \dim \mathscr{O}_{X,x}$. But the rank of the Jacobian matrix does not change when we replace x by $\overline{x}$ (and consider the polynomials g_i in $K[X_\bullet]$), and $\dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{X_K,\overline{x}}$. Since $\overline{x}$ is a regular point of X_K by (iii), which we now assume to hold, we are done. □

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Corollary 2.22. *Let X be an irreducible scheme of finite type over k, and let $x \in X(k)$ be a k-valued point. Then $X \to \operatorname{Spec} k$ is smooth at x if and only if $\dim X = \dim_k T_x X$.*

Corollary 2.23. *Let $X = V(g_1, \dots, g_s) \subseteq \mathbb{A}^n_k$ and let $x \in X$ be a smooth closed point. Let $d = \dim \mathscr{O}_{X,x}$. Then $J_{g_1,\dots,g_s}(x)$ has rank $n-d$. In particular, $s \geq n - d$.*

After renumbering the g_i, if necessary, there exists an open neighborhood U of x and an open immersion $U \subseteq V(g_1, \dots, g_{n-d})$, i.e., locally around x, "X is cut out in affine space by the expected number of equations".

Corollary 2.24. *Let X be locally of finite type over k. The following are equivalent:*
(i) *X is smooth over k.*
(ii) *$X \otimes_k L$ is regular for every field extension L/k.*

(iii) *There exists an algebraically closed extension field K of k such that $X \otimes_k K$ is regular.*

The sheaf of differentials.

General references: [GW2] Ch. 17, [M2] §25, [Bo] Ch. 8, [H] II.8.

We now introduce the "module of differentials" of a ring homomorphism (and its sheaf version $\Omega_{X/S}$ for a scheme morphism $X \to S$). This allows us to study how the (co-)tangent space varies in a family; as we will see, under suitable assumptions the fiber $\Omega_{X/S}(x)$ at $x \in X$ can be identified with the dual $\mathfrak{m}_x/\mathfrak{m}_x^2$ of T_xX, see Proposition 2.44. The theory we will set up is also closely related to the so-called infinitesimal lifting criterion for smooth morphisms, see Theorem 2.54.

(2.7) Modules of differentials.

Let A be a ring.

Definition 2.25. *Let B be an A-algebra, and M a B-module. An A-*derivation *from B to M is a homomorphism $D\colon B \to M$ of abelian groups such that*
(a) *(Leibniz rule) $D(bb') = bD(b') + b'D(b)$ for all $b, b' \in B$,*
(b) *$d(a) = 0$ for all $a \in A$.*

Assuming property (a), property (b) is equivalent to saying that D is a homomorphism of A-modules. We denote the set of A-derivations $B \to M$ by $\mathrm{Der}_A(B, M)$; it is naturally a B-module.

Definition 2.26. *Let B be an A-algebra. We call a B-module $\Omega_{B/A}$ together with an A-derivation $d_{B/A}\colon B \to \Omega_{B/A}$ a module of (relative, Kähler) differentials of B over A if it satisfies the following universal property:*

For every B-module M and every A-derivation $D\colon B \to M$, there exists a unique B-module homomorphism $\psi\colon \Omega_{B/A} \to M$ such that $D = \psi \circ d_{B/A}$.

In other words, the map $\mathrm{Hom}_B(\Omega_{B/A}, M) \to \mathrm{Der}_A(B, M)$, $\psi \mapsto \psi \circ d_{B/A}$ is a bijection.

Lemma 2.27. *Let I be a set, $B = A[T_i, i \in I]$ the polynomial ring. Then $\Omega_{B/A} := B^{(I)}$ with $d_{B/A}(T_i) = e_i$, the "i-th standard basis vector" is a module of differentials of B/A.*

So we can write $\Omega_{B/A} = \bigoplus_{i \in I} Bd_{B/A}(T_i)$.

Lemma 2.28. *Let $\varphi\colon B \to B'$ be a surjective homomorphism of A-algebras, and write $\mathfrak{b} = \mathrm{Ker}(\varphi)$. Assume that a module of differentials $(\Omega_{B/A}, d_{B/A})$*

for B/A exists. Then

$$\Omega_{B/A}/(\mathfrak{b}\Omega_{B/A} + B'd(\mathfrak{b}))$$

together with the derivation $d_{B'/A}$ induced by $d_{B/A}$ is a module of differentials for B'/A.

Corollary 2.29. *For every A-algebra B, a module $\Omega_{B/A}$ of differentials exists. It is unique up to unique isomorphism.*

We will see later that for a scheme morphism $X \to Y$, one can construct an $\mathscr{O}_X$-module $\Omega_{X/Y}$ together with a "derivation" $\mathscr{O}_X \to \Omega_{X/Y}$ by gluing sheaves associated to modules of differentials attached to the coordinate rings of suitable affine open subschemes of X and Y.

Let $\varphi\colon A \to B$ be a ring homomorphism. For the next definition, we will consider the following situation: Let C be a ring, $I \subseteq C$ an ideal with $I^2 = 0$, and let

$$\begin{tikzcd} C/I & B \arrow[l] \\ C \arrow[u] & A \arrow[l] \arrow[u, "\varphi"'] \end{tikzcd}$$

be a commutative diagram (where the right vertical arrow is the canonical projection). We will consider the question whether for these data, there exists a homomorphism $B \to C$ (dashed in the following diagram) making the whole diagram commutative:

$$\begin{tikzcd} C/I & B \arrow[l] \arrow[dl, dashed] \\ C \arrow[u] & A. \arrow[l] \arrow[u, "\varphi"'] \end{tikzcd}$$

Definition 2.30. *Let $\varphi\colon A \to B$ be a ring homomorphism.*

(1) *We say that φ is* formally unramified*, if in every situation as above, there exists at most one homomorphism $B \to C$ making the diagram commutative.*
(2) *We say that φ is* formally smooth*, if in every situation as above, there exists at least one homomorphism $B \to C$ making the diagram commutative.*
(3) *We say that φ is* formally étale*, if in every situation as above, there exists a unique homomorphism $B \to C$ making the diagram commutative.*

Passing to the spectra of these rings, we can interpret the situation in geometric terms: $\operatorname{Spec} C/I$ is a closed subscheme of $\operatorname{Spec} C$ with the same topological space, so we can view the latter as an "infinitesimal thickening" of the former. The question becomes the question whether we can extend the morphism from $\operatorname{Spec} C/I$ to $\operatorname{Spec} B$ to a morphism from this thickening.

Construction 2.31. Let B be a ring and let M be a B-module. We construct a B-algebra $D_B(M)$ as follows. As additive groups, we set $D_B(M) = B \times M$. The multiplication is defined by

$$(b, m)(b', m') = (bb', bm' + b'm).$$

Then $M = \{0\} \times M \subseteq D_B(M)$ is an ideal with $M^2 = 0$.

For example, taking $M = B$, we have $D_B(B) \cong B[\varepsilon](\varepsilon^2)$, the ring of dual numbers over B.

The projection $\pi\colon D_B(M) \to B$ is a B-algebra homomorphism, i.e., the composition $B \to D_B(M) \to B$ is the identity.

Now suppose that B is an A-algebra. One then checks that the map

$$\mathrm{Der}_A(B, M) \to \{\psi \in \mathrm{Hom}_A(B, D_B(M));\ \pi \circ \psi = \mathrm{id}_B\}, \quad D \mapsto (b \mapsto (b, D(b)),$$

is a B-module isomorphism.

Proposition 2.32. *Let $\varphi\colon A \to B$ be a ring homomorphism. Then φ is formally unramified if and only if $\Omega_{B/A} = 0$.*

Proof. Assume that $\Omega_{B/A} = 0$, and consider $I \subset C$ and a commutative diagram as above. We need to show that there is at most one ring homomorphism $B \to C$ making the diagram commutative. Assume that $\varphi_1, \varphi_2\colon B \to C$ have this property. The C-module structure on I factors through a C/I-module structure since $I^2 = 0$, so that we can view I as a B-module via the map $B \to C/I$. Then the difference $\varphi_1 - \varphi_2$ is an A-derivation $B \to I$, and is hence zero by our assumption.

For the converse it is enough that every A-derivation $B \to M$ vanishes. Let $C = D_B(M)$ and $I = M$. Then $I^2 = 0$, and the assumption that B is formally unramified over A implies $\mathrm{Der}_A(B, M) = 0$. $\square$

For an algebraic field extension L/K one can show that $K \to L$ is formally unramified if and only if it is formally smooth if and only if L/K is separable. Cf. Problem 27 and [M2] §25, §26 (where the discussion is extended to the general, not necessarily algebraic, case).

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Theorem 2.33. *Let $f\colon A \to B$, $g\colon B \to C$ be ring homomorphisms.*

(1) *Then we obtain a natural sequence of C-modules*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

which is exact.

(2) *If moreover g is formally smooth, then the sequence*

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

is a split short exact sequence.

(3) *Conversely, assume that $g \circ f$ is formally smooth and that the sequence*

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

is a split short exact sequence. Then g is formally smooth.

Proof. To check the exactness in Part (1), it is enough to check that the sequence gives rise to an exact sequence whenever we apply the functor $\mathrm{Hom}_C(-, M)$ for M a C-module. Note that $\mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, M) = \mathrm{Hom}_B(\Omega_{B/A}, M)$ (where on the right we view M as a B-module via g). See [ALG2] Satz 3.14.

Thus the first part follows, once we check that

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M)$$

is exact (as a sequence of A-modules or just abelian groups) for any C-module M. But this is obvious.

Part (2). Now assume that g is formally smooth. Let us construct a C-module homomorphism $\Omega_{C/A} \to \Omega_{B/A} \otimes_B C$ as follows. Constructing a homomorphism like this amounts to constructing an A-derivation $C \to \Omega_{B/A} \otimes_B C =: M$. Similarly as above, we consider $C \times M$ as a ring (with $M^2 = 0$). Let $B \to C \times M$ be given by $b \mapsto (g(b), db \otimes 1)$. One checks that this is a ring homomorphism. Since g is formally smooth, for this B-algebra structure we find a homomorphism $C \to C \times M$ of B-algebras. Composing it with the projection to M we obtain an A-derivation $C \to M$. One checks that the composition $\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{B/A} \otimes_B C$ is the identity, and this finishes the proof.

See also [GW2] Proposition 18.18 (1) for a slightly different proof of the second part (which is more along the lines of our proof of the first part).

Part (3). This can be proved by similar arguments as for Parts (1) and (2). We omit the proof for the time being (see [GW2] Proposition 18.18 (2)). □

Theorem 2.34. *Let $f\colon A \to B$, $g\colon B \to C$ be ring homomorphisms. Assume that g is surjective with kernel $\mathfrak{b}$.*

(1) *There is a natural sequence of C-modules*

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0,$$

where the homomorphism $\mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B C$ is given by $x \mapsto d_{B/A}(x) \otimes 1$.

(2) *If moreover $g \circ f$ is formally smooth, then the sequence*

$$0 \to \mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

is a split short exact sequence.

Proof. All assertions in Part (1) follow from Theorem 2.33 and Lemma 2.28.

To prove Part (2), consider the short exact sequence

$$0 \to \mathfrak{b}/\mathfrak{b}^2 \to B/\mathfrak{b}^2 \xrightarrow{p} C \to 0.$$

The assumption that $g \circ f$ is formally smooth implies that p admits a section s. Then $s \circ p_{|\mathfrak{b}/\mathfrak{b}^2} = 0$, and $p \circ (\mathrm{id} - s \circ p) = 0$. We obtain $D := \mathrm{id} - s \circ p\colon B/\mathfrak{b}^2 \to \mathfrak{b}/\mathfrak{b}^2$. This is an element of $\mathrm{Der}_A(B/\mathfrak{b}^2, \mathfrak{b}/\mathfrak{b}^2) = \mathrm{Hom}_B(\Omega_{B/A}, \mathfrak{b}/\mathfrak{b}^2)$ and one checks that it gives rise to a retraction of the map $\mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B C$ in the sequence in Part (2). □

(2.8) The sheaf of differentials of a scheme morphism.

Remark 2.35. Let again B an A-algebra. There is the following alternative construction of $\Omega_{B/A}$: Let $m\colon B \otimes_A B \to B$ be the multiplication map, and let $I = \mathrm{Ker}(m)$. Then I/I^2 is a B-module, and $d\colon B \to I/I^2$, $b \mapsto 1\otimes b - b\otimes 1$, is an A-derivation.

Let us show that $(I/I^2, d)$ satisfies the universal property defining $(\Omega_{B/A}, d_{B/A})$. Let M be a B-module. Composition with d gives a map $\mathrm{Hom}_A(I/I^2, M) \to \mathrm{Der}_A(B, M)$. To show that it is injective, it is enough to show that I/I^2 is generated by the image of d as a B-module. This follows from the following two computations (for $b, b', b_i, b'_i \in B$):
(1) $b \otimes b' = bb' \otimes 1 + (b \otimes 1)(1 \otimes b' - b' \otimes 1)$,
(2) if $\sum b_i b'_i = 0$, then $\sum b_i \otimes b'_i = \sum (b_i \otimes 1)(1 \otimes b'_i - b'_i \otimes 1)$ by (1).

For the surjectivity, let $D \in \mathrm{Der}_A(B, M)$ and let $\psi\colon B \to D_B(M)$, $b \mapsto (b, D(b))$, the corresponding map, cf. Construction 2.31. The diagram

$$\begin{tikzcd} 0 \arrow[r] & I/I^2 \arrow[r] & B \otimes B/I^2 \arrow[r] \arrow[d, "b\otimes b' \mapsto b\psi(b')"] & B \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & M \arrow[r] & D_B(M) \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

(with exact rows) induces a map $\delta\colon I/I^2 \to M$ which makes the whole diagram commute, and $\delta \circ d = D$.

There are several ways of defining an $\mathscr{O}_X$-module of differentials for a morphism $f\colon X \to Y$ of schemes. One way is to proceed by gluing, using the following remark. Nov. 8, 2023

Remark 2.36. Let $A \to B$ be a ring homomorphism, and let $S \subseteq B$ be a multiplicative subset. Then there is a natural identification $S^{-1}\Omega_{B/A} = \Omega_{N/A} \otimes_B S^{-1}B = \Omega_{S^{-1}/A}$. If $T \subseteq A$ is a multiplicative subset that is mapped to $(S^{-1}B)^\times$ under the natural homomorphism $A \to B \to S^{-1}B$, then this module can also be identified with $\Omega_{S^{-1}B/T^{-1}A}$.

To pin down the sheaf of differentials we first define the notion of derivation in this context.

Definition 2.37. *Let $X \to Y$ be a morphism of schemes, and let $\mathscr{M}$ be an $\mathscr{O}_X$-module. A derivation $D\colon \mathscr{O}_X \to \mathscr{M}$ is a homomorphism of abelian sheaves such that for all open subsets $U \subseteq X$, $V \subseteq Y$ with $f(U) \subseteq V$, the map $\mathscr{O}(U) \to \mathscr{M}(U)$ is an $\mathscr{O}_Y(V)$-derivation.*

Equivalently, $D\colon \mathscr{O}_X \to \mathscr{M}$ is a homomorphism of $f^{-1}(\mathscr{O}_Y)$-modules such that for every open $U \subseteq X$, the Leibniz rule

$$D(U)(bb') = bD(U)(b') + b'D(U)(b), \qquad \forall b, b' \in \Gamma(U, \mathscr{O}_X)$$

holds.

We denote the set of all these derivations by $\mathrm{Der}_Y(\mathscr{O}_X, \mathscr{M})$; it is a $\Gamma(X, \mathscr{O}_X)$-module.

Definition/Proposition 2.38. *Let $f\colon X \to Y$ be a morphism of schemes. The following three definitions give the same result (up to unique isomorphism), called the sheaf of differentials of f or of X over Y, denoted $\Omega_{X/Y}$ — a quasi-coherent $\mathscr{O}_X$-module together with a derivation $d_{X/Y}\colon \mathscr{O}_X \to \Omega_{X/Y}$.*

(i) *There exists a unique $\mathscr{O}_X$-module $\Omega_{X/Y}$ together with a derivation $d_{X/Y}\colon \mathscr{O}_X \to \Omega_{X/Y}$ such that for all affine open subsets $\operatorname{Spec} B = U \subseteq X$, $\operatorname{Spec} A = V \subseteq Y$ with $f(U) \subseteq V$, $\Omega_{X/Y|U} = \widetilde{\Omega_{B/A}}$ and $d_{X/Y|U}$ is induced by $d_{B/A}$.*

(ii) *Define $\Omega_{X/Y} = \Delta^*(\mathscr{J}/\mathscr{J}^2)$, where $\Delta\colon X \to X \times_Y X$ is the diagonal morphism, $W \subseteq X \times_Y X$ is open such that $\operatorname{Im}(\Delta) \subseteq W$ is closed (if f is separated we can take $W = X \times_Y X$), and $\mathscr{J}$ is the quasi-coherent ideal defining the closed subscheme $\Delta(X) \subseteq W$. Define the derivation $d_{X/Y}$ as the one induced, on affine opens, by the map $b \mapsto 1 \otimes b - b \otimes 1$.*

(iii) *The quasi-coherent $\mathscr{O}_X$-module $\Omega_{X/Y}$ together with $d_{X/Y}$ is characterized by the universal property that composition with $d_{X/Y}$ induces bijections*

$$\operatorname{Hom}_{\mathscr{O}_X}(\Omega_{X/Y}, \mathscr{M}) \xrightarrow{\sim} \operatorname{Der}_Y(\mathscr{O}_X, \mathscr{M})$$

for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{M}$, functorially in $\mathscr{M}$.

The properties we proved for modules of differentials can be translated into statements for sheaves of differentials. In all statements here, equality means that there is a unique isomorphism that is compatible with the universal derivations.

Proposition 2.39. *Let $f\colon X \to Y$ be a morphism of schemes.*

(1) *Let $g\colon Y' \to Y$ be a morphism of schemes, and let $X' = X \times_Y Y'$. Denote by $g'\colon X' \to X$ the base change of g. There is a natural isomorphism $\Omega_{X'/Y'} = (g')^*\Omega_{X/Y}$.*

(2) *Let $U \subseteq X$ and $V \subseteq V$ be open subsets with $f(U) \subseteq V$. There is a natural identification $\Omega_{X/Y|U} = \Omega_{U/V}$.*

(3) *Let $x \in X$. Then $\Omega_{X/Y,x} = \Omega_{\mathscr{O}_{x,x}/\mathscr{O}_{Y,y}}$.*

We can use a similar definition as we used for ring homomorphisms above to define the notions of formally unramified, formally smooth and formally étale morphisms of schemes.

Definition 2.40. *Let $f\colon X \to Y$ be a morphism of schemes.*

(1) *We say that f is* formally unramified*, if for every ring C, every ideal I with $I^2 = 0$, and every morphism $\operatorname{Spec} C \to Y$ (which we use to view $\operatorname{Spec} C$ and $\operatorname{Spec} C/I$ as Y-schemes), the composition with the natural closed embedding $\operatorname{Spec} C/I \to \operatorname{Spec} C$ yields an injective map $\operatorname{Hom}_Y(\operatorname{Spec} C, X) \to \operatorname{Hom}_Y(\operatorname{Spec} C/I, X)$.*

(2) *We say that f is* formally smooth*, if for every ring C, every ideal I with $I^2 = 0$, and every morphism $\operatorname{Spec} C \to Y$, the composition with the natural closed embedding $\operatorname{Spec} C/I \to \operatorname{Spec} C$ yields a surjective map $\operatorname{Hom}_Y(\operatorname{Spec} C, X) \to \operatorname{Hom}_Y(\operatorname{Spec} C/I, X)$.*

(3) *We say that f is* formally étale, *if f is formally unramified and formally smooth.*

If f is a morphism of affine schemes, then f has one of the properties of this definition if and only if the corresponding ring homomorphism has the same property in the sense of our previous definition.

Proposition 2.41. *Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms of schemes. Then there is an exact sequence*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

of $\mathscr{O}_X$-modules. If f is formally smooth, then the sequence

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

is exact and splits locally on X.

Proposition 2.42. *Let $i\colon Z \to X$ be a closed immersion with corresponding ideal sheaf $\mathscr{J} \subseteq \mathscr{O}_X$, and let $g\colon X \to Y$ be a scheme morphism. Then there is an exact sequence*

$$i^*(\mathscr{J}/\mathscr{J}^2) \to i^*\Omega_{X/Y} \to \Omega_{Z/Y} \to 0$$

of $\mathscr{O}_Z$-modules. If Z is formally smooth over Y, then the sequence

$$0 \to i^*(\mathscr{J}/\mathscr{J}^2) \to i^*\Omega_{X/Y} \to \Omega_{Z/Y} \to 0$$

is exact and splits locally on Z.

Remark 2.43. When we say that a short exact sequence of $\mathscr{O}_X$-modules *splits locally on a scheme X*, this means that there exists an open cover $X = \bigcup_i U_i$ such that for each i the sequence splits after restricting it to U_i. (It follows from the short exact sequence attached to the local-to-global spectral sequence for Ext sheaves and the vanishing of higher cohomology of quasi-coherent sheaves on affines that a short exact sequence of quasi-coherent $\mathscr{O}_X$-modules that splits locally on X and where the term on the right hand side is of finite presentation, splits on every affine open of X.)

Applying Proposition 2.42 to X a scheme of finite type over $Y = \operatorname{Spec}(k)$, k a field, and $Z = \operatorname{Spec}(k)$ so that i is a k-valued point, we obtain the following description of the fiber of the sheaf of differentials at x.

Proposition 2.44. *Let K be a field, and let X be a k-scheme of finite type. Let $x \in X(k)$. Then we have an isomorphism $T_xX = \Omega_{X/k}(x)^\vee$ between the Zariski tangent space at x and the dual space of the fiber of the sheaf of differentials of X/k at x.*

Similarly, we have the following description. For any scheme Y, we write $Y[\varepsilon] := Y \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon]/(\varepsilon^2)$. Denote by $\iota_Y\colon Y \to Y[\varepsilon]$ the natural map. For any morphism $X \to S$ of schemes we write Nov. 13, 2023

$$\mathscr{T}_{X/S} := \mathscr{H}om_{\mathscr{O}_X}(\Omega_{X/S}, \mathscr{O}_X)$$

and call this the *tangent sheaf* of X over S.

Proposition 2.45. *Let $f\colon X \to S$ be a morphism of schemes. For every X-scheme $g\colon Y \to X$, we have a bijection*

$$\mathrm{Hom}^g(Y[\varepsilon], X) := \{\tilde{g}\colon Y[\varepsilon] \to X;\ \tilde{g} \circ \iota_Y = g\} \xrightarrow{\cong} \Gamma(Y, g^*\mathscr{T}_{X/Y}),$$

and these bijections are functorial in Y.

Proof. First note that $\mathrm{Hom}^g(Y[\varepsilon], X)$ can be identified with $\mathrm{Der}_S(\mathscr{O}_X, g_*\mathscr{O}_Y)$. In fact, this can be checked on an affine open cover, and in the affine case we have seen this in Construction 2.31. Now we conclude by the following chain of isomorphisms:

$$\mathrm{Der}_S(\mathscr{O}_X, g_*\mathscr{O}_Y) = \mathrm{Hom}_{\mathscr{O}_X}(\Omega_{X/S}, g_*\mathscr{O}_Y) = \mathrm{Hom}_{\mathscr{O}_Y}(g^*\Omega_{X/S}, \mathscr{O}_Y) = \Gamma(Y, g^*\mathscr{T}_{X/S}).$$

□

Using this description, we "compute" the sheaf of differentials of projective space.

Proposition 2.46. *Let R be a ring. We have a short exact sequence*

$$0 \to \Omega_{\mathbb{P}^n_R/R} \to \mathscr{O}(-1)^{n+1} \to \mathscr{O} \to 0$$

of $\mathscr{O}_X$-modules, called the Euler sequence*.*

Proof. Write $X = \mathbb{P}^n_R$. We have the "universal" surjection $\mathscr{O}_X^{n+1} \to \mathscr{O}_X(1)$ and denote by $\mathscr{K}$ its kernel. We want to show that $\mathscr{K}(-1) := \mathscr{K} \otimes_{\mathscr{O}_X} \mathscr{O}_X(-1) \cong \Omega_{X/R}$. All the $\mathscr{O}_X$-modules involved here are locally free of finite rank, so it is enough to prove that $\mathscr{T}_{X/R} \cong \mathscr{H}om(\mathscr{K}, \mathscr{O}(1)) = \mathscr{H}om(\mathscr{K}(-1), \mathscr{O}) = \mathscr{K}(-1)^\vee$.

Let $U = \mathrm{Spec}(A) \subseteq X$ be open affine and denote by $g\colon U \to X$ the inclusion. The morphism g corresponds to a surjection $A^{n+1} \to L$ onto a locally free A-module L of rank 1 whose kernel we denote by K. Note that $K = \mathscr{K}_{|U}$. We now use the notation of Proposition 2.45.

Claim. There is a natural identification $\mathrm{Hom}_A(K, A^{n+1}/K) \xrightarrow{\cong} \mathrm{Hom}^g(U[\varepsilon], X)$.

Proof of claim. An element of $\mathrm{Hom}^g(U[\varepsilon], X) \subseteq X(U(\varepsilon))$ is given by a surjection $(A[\varepsilon]/(\varepsilon^2))^{n+1} \to L'$, where L' is locally free over $A[\varepsilon]/(\varepsilon^2)$ of rank 1, or equivalently its kernel $K' \subset (A[\varepsilon]/(\varepsilon^2))^{n+1}$, such that $K' \otimes_{A[\varepsilon]/(\varepsilon^2)} A = K$.

Now take an A-module homomorphism $\alpha\colon K \to A^{n+1}/K$. We define K' as the $A[\varepsilon]/(\varepsilon^2)$-module generated by the image of the map $K \to (A[\varepsilon]/(\varepsilon^2))^{n+1}$, $x \to x + \varepsilon\alpha(x)$. (We define $\varepsilon\alpha(x)$ by choosing a lift of $\alpha(x) \in A^{n+1}/K$ in A^{n+1}. The resulting K' is independent of the choice of lift.)

Note that $A[\varepsilon]^{n+1}/K'$ is locally free over $A[\varepsilon]$. To check this, we may localize and thus assume that K and A^{n+1}/K are free A-modules. Now choosing lifts of bases of K and A^{n+1}/K to $A[\varepsilon]^{n+1}$ gives us a family of $n+1$ vectors. Write them as the columns of a matrix M over $A[\varepsilon]$. By construction, $\det(M)$ maps to a unit in A and hence is a unit in $A[\varepsilon]$. Thus the lifts form a basis of $A[\varepsilon]^{n+1}$ and in particular $A[\varepsilon]^{n+1}/K'$ (and K') are free.

This defines the desired bijection.

With the claim and Proposition 2.45 we can identify $\Gamma(U, \mathscr{T}_{X/R})$ with

$$\mathrm{Hom}_A(K, A^{n+1}/K) = \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{K}_{|U}, \mathscr{O}_X(1)_{|U}) = \Gamma(U, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{K}, \mathscr{O}_X(1))).$$

This identification is compatible with restrictions to smaller subsets and therefore defines the isomorphism of $\mathscr{O}_X$-modules we are looking for. □

Remark 2.47.

(1) In the course of the proof we have established a canonical identification of the tangent space $T_x\mathbb{P}^n_k$ of projective space over a field k in a k-valued point x with the vector space $\mathrm{Hom}_k(K, k^{n+1}/K)$, where $K = \mathrm{Ker}(k^{n+1}, L)$ is the kernel of the quotient of k^{n+1} corresponding to x via the functorial description of $\mathbb{P}^n_k$. At this point we use the point of view that $\mathbb{P}^n_k(k)$ is the set of all 1-dimensional quotients of k^{n+1}, or equivalently – passing to the kernel of the projection – of all hyperplanes in k^{n+1}.

Passing to the dual (and classical) point of view, K gives us a line $K^\perp = (k^{n+1}/K)^\vee$ in the dual vector space $(k^{n+1})^\vee$ (which we could identify with k^{n+1} via the standard basis). Then the tangent space is identified with $\mathrm{Hom}_k(K^\perp, k^{n+1,\vee}/K^\perp)$, which is isomorphic to $k^{n+1,\vee}/K^\perp$ since $K^\perp \cong k$. This is "the same" description as using the natural surjection $\mathbb{A}^{n+1}_k \setminus \{0\} \to \mathbb{P}^n_k$ which induces surjections on tangent spaces, cf. [GW1] Prop. 6.10.

(2) As for every short exact sequence of locally free modules of finite rank, we obtain an identification for the top exterior powers,

$$\bigwedge^n \Omega_{\mathbb{P}^n_R/R} \cong \bigwedge^n \Omega_{\mathbb{P}^n_R/R} \otimes \bigwedge^1 \mathscr{O}_{\mathbb{P}^n_R/R} \cong \bigwedge^{n+1} \mathscr{O}(-1)^{n+1} \cong \mathscr{O}(-n-1).$$

Example 2.48. For $n = 1$, the previous proposition gives $\Omega_{\mathbb{P}^1_R/R} \cong \mathscr{O}_{\mathbb{P}^1_R}(-2)$. This is also easy to check directly. The key computation is

$$0 = d(1) = d(\frac{X_i}{X_j}\frac{X_j}{X_i}) = \frac{X_i}{X_j}d\frac{X_j}{X_i} + \frac{X_j}{X_i}d\frac{X_i}{X_j},$$

which implies

$$d\frac{X_j}{X_i} = -\left(\frac{X_j}{X_i}\right)^2 d\frac{X_i}{X_j}.$$

The latter equality describes how $\Omega_{\mathbb{P}^1_R/R}$ is glued from the free modules $\Omega_{\mathbb{P}^1_R/R|D_+(X_i)}$. It coincides with the way we glue to obtain $\mathscr{O}(-2)$.

(2.9) Sheaves of differentials and smoothness.

We start by slightly rephrasing the definition of a smooth morphism. Nov. 15, 2023

Definition 2.49. *A morphism $f\colon X \to Y$ of schemes is called* smooth of relative dimension $d \geq 0$ in $x \in X$, *if there exist affine open neighborhoods $U \subseteq X$ of x and $V = \operatorname{Spec} R \subseteq Y$ of $f(x)$ such that $f(U) \subseteq V$ and an open immersion $j\colon U \to \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d})$ such that the triangle*

$$\begin{tikzcd} U \arrow[rr, "j"] \arrow[dr, "f"'] & & \operatorname{Spec} R[T_1, \dots, T_n]/(f_1, \dots, f_{n-d}) \arrow[dl] \\ & V & \end{tikzcd}$$

is commutative, and that the images of $df_1, \dots, df_{n-d}$ in the fiber $\Omega^1_{\mathbb{A}^n_R/R} \otimes \kappa(x)$ are linearly independent over $\kappa(x)$. (We view x as a point of $\mathbb{A}^n_R$ via the embedding $U \to \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d}) \to \operatorname{Spec} R[T_1, \dots, T_n] = \mathbb{A}^n_R$.)

To see the equivalence, use that $df = \sum_i \frac{\partial f}{\partial X_i} dX_i$.

Proposition 2.50. *Let $f\colon X \to Y$ be smooth of relative dimension d at $x \in X$. Then there exists an open neighborhood U of x such that the restriction $\Omega_{X/Y|U} (= \Omega_{U/Y})$ is free of rank d.*

Proof. Since the assertion is local on X, we may assume that $Y = \operatorname{Spec} R$ and $X = \operatorname{Spec} R[T_1, \dots, T_n]/(f_1, \dots, f_{n-d})$ with the $df_i(x) \in \Omega_{\mathbb{A}^n_R/R}(x)$ linearly independent, as in Definition 2.49. We write $\mathfrak{a} = (f_1, \dots, f_{n-d})$ and $A = R[T_1, \dots, T_n]/\mathfrak{a}$. We have the exact sequence (Theorem 2.34)

$$\mathfrak{a}/\mathfrak{a}^2 \to \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A \to \Omega_{A/R} \to 0.$$

Renumber the X_i (if necessary) so that the images of $dX_1, \dots, dX_d, df - 1, \dots, df_{n-d}$ are a basis of the fiber $(\Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A)(x)$ over x. By the lemma of Nakayama, these elements give us also a basis of the stalk, and hence even a basis on an open neighborhood U of x. The image of $\mathfrak{a}/\mathfrak{a}^2$ is exactly the submodule generated by the df_i, so this implies that $\Omega_{A/R}$ is free over such a neighborhood. □

Remark 2.51. Let us check that in the situation of the previous proposition (and with the notation of its proof), the sequence

$$0 \to \mathfrak{a}/\mathfrak{a}^2 \to \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A \to \Omega_{A/R} \to 0.$$

is split exact over U. Since $(\Omega_{A/R})_{|U}$ is free, it is clear that the sequence splits, once we have shown the exactness. Thus it is enough to show that the map on the left hand side is injective. Take $g = \sum c_j f_j \in \mathfrak{a}$, $c_i \in R[T_\bullet]$. Then

$$dg = \sum_j (c_j df_j + f_j dc_j) = \sum_j c_j df_j \in \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A = \Omega_{R[T_\bullet]/R}/\mathfrak{a},$$

so if $dg = 0$, then (after restricting to U, where the df_j are part of a basis) all c_j lie in $\mathfrak{a}$ and hence $g \in \mathfrak{a}^2$.

Theorem 2.52. *Let k be an algebraically closed field, and let X be an irreducible k-scheme of finite type. Let $d = \dim X$. Then X is smooth over k if and only if $\Omega_{X/k}$ is locally free of rank d.*

Proof. If $\Omega_{X/Y}$ is locally free of rank $\dim X$, then X is regular and hence, since k is algebraically closed, also smooth over k (Theorem 2.21). Conversely, the smoothness of f implies that $\Omega_{X/Y}$ is locally free by Proposition 2.50. Again using Theorem 2.21, we also obtain that X is regular, and it follows that f must be smooth of relative dimension $\dim X$. □

Proposition 2.53. *Let $f\colon X \to Y$ be smooth of relative dimension d at $x \in X$. Then there exists an open neighborhood U of x such that the restriction $U \to Y$ of f to U is formally smooth.*

Proof. As in the proof of Proposition 2.50, it is enough to consider the local situation, and we again use the notation set up in the beginning of the proof of that proposition.

Consider a ring C, an ideal I of C with $I^2 = 0$ and a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & C/I \\ \uparrow & & \uparrow \\ R & \longrightarrow & C. \end{array}$$

We need to show that there exists a homomorphism $\varphi\colon A \to C$ making the diagram commutative. We start by choosing arbitarily an R-algebra homomorphism $\psi\colon R[T_1,\dots,T_n] \to C$ such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & C/I \\ \uparrow & & \uparrow \\ R[T_\bullet] & \longrightarrow & C \end{array}$$

is commutative. Then $\psi(\mathfrak{a}) \subseteq I$ (but of course there is no reason to expect that ψ will factor through A; we will now change it appropriately to achieve that). In Remark 2.51 we have seen that the sequence

$$0 \to \mathfrak{a}/\mathfrak{a}^2 \to \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A \to \Omega_{A/R} \to 0.$$

is split exact, at least after replacing A by a suitable localization. Since the proposition makes only a local statement the localization is harmless and we suppress it from the notation. The restriction of ψ to $\mathfrak{a}$ induces a map $\mathfrak{a}/\mathfrak{a}^2 \to I/I^2 = I$, as we have already noted, and since the sequence is split, we can extend that map to a map $\xi\colon \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A \to I$. We define D as the composition

$$R[T_\bullet] \xrightarrow{d} \Omega_{R[T_\bullet]/R} \to \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A \xrightarrow{\xi} I,$$

an R-derivation with the property that $\psi_{|\mathfrak{a}} = D_{|\mathfrak{a}}$. Setting $\varphi = \psi - D$, we obtain a map that maps $\mathfrak{a}$ to 0 and (since D is a derivation) is a ring homomorphism. Thus φ factors through a homomorphism $\varphi\colon A \to C$. This makes the above diagram commutative, so we are done. $\square$

Theorem 2.54. *Let $f\colon X \to Y$ be a morphism locally of finite presentation (e.g., if Y is noetherian and f is locally of finite type). Then f is smooth if and only if f is formally smooth.*

Proof. Let f be formally smooth and locally of finite presentation. To show that f is smooth, we may work locally on X and Y and therefore pass to an affine situation, i.e., assume that f is given by a ring homomorphism $R \to R[T_\bullet] \to A$ with $R[T_\bullet] \to A$ surjective with kernel $\mathfrak{a}$. Then Theorem 2.34 shows that the sequence

$$0 \to \mathfrak{a}/\mathfrak{a}^2 \to \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A \to \Omega_{A/R} \to 0.$$

is split exact. Choosing a basis of $\mathfrak{a}/\mathfrak{a}^2$ and lifting its elements to polynomials $f_1, \dots, f_{n-d} \in \mathfrak{a}$, we see that the conditions of Definition 2.49 are satisfied and the morphism $\operatorname{Spec} A \to \operatorname{Spec} R$ is smooth.

For the converse, note that the previous proposition shows already that a smooth morphism is at least "locally formally smooth". We only give some very sketchy indications on how to get a global version. See [Bo] Ch. 8.5 for more details. See also [GW2] Section (18.10) for a slightly different approach.

Consider a diagram

$$\begin{tikzcd} \operatorname{Spec} C/I \arrow[r, "a_0"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} C \arrow[r] & Y \end{tikzcd}$$

where, as usual, $I \subseteq C$ is an ideal with $I^2 = 0$. We have seen that there exists an open cover $X = \bigcup_i U_i$ such that each U_i is formally smooth over Y. In particular after restricting a_0 to U_i and the inverse image of U_i in $\operatorname{Spec} C/I$, we can find the desired diagonal morphism that extends a_0 to $\operatorname{Spec} C$. In other words, we find an open cover $(V_i)_i$ of $\operatorname{Spec} C$ (which topologically is $= \operatorname{Spec} C/I$) and morphisms $\varphi_i\colon V_i \to U_i \subseteq X$ making the above diagram commutative. The idea is to replace the φ_i by φ'_i such that φ'_i and φ'_j coincide on $V_i \cap V_j$. By gluing one obtains the desired map $\operatorname{Spec} C \to X$.

Here, we want to set $\varphi'_i = \varphi_i - D_i$ for some derivation D_i (cf. Construction 2.31 where we have seen this principle). Writing this out one sees that there exists a family $(D_i)_i$ with the desired properties if and only if a certain class (depending on the φ_i) in $\check{H}^1(\operatorname{Spec} C/I, \mathscr{H}om_{\mathscr{O}_{\operatorname{Spec} C/I}}(a_0^*\Omega_{X/Y}, \tilde{I})$ vanishes. But this cohomology group vanishes entirely since $\operatorname{Spec} C/I$ is affine and $\mathscr{H}om_{\mathscr{O}_{\operatorname{Spec} C/I}}(a_0^*\Omega_{X/Y}, \tilde{I})$, $\Omega_{X/Y}$ being of finite presentation by our assumptions, is quasi-coherent. $\square$

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Remark 2.55. We mention the following further facts without proof. See for instance [GW2] Chapter 18.

(1) A morphism of schemes is smooth if and only if it is locally of finite presentation, flat and has regular geometric fibers. (Here the geometric fibers of a morphism $X \to Y$ are the schemes $X \times_Y \operatorname{Spec}(K)$ where K is an algebraically closed field and the fiber product is taken with respect to f and a K-valued point of Y.) This also gives us a “fiber criterion for smoothness”, cf. [GW2] Corollary 18.77.
(2) A morphism of schemes is étale (which we have defined as smooth of relative dimension 0) if and only if it is locally of finite presentation and formally étale if and only if it is flat and unramified.
(3) An étale morphism is locally standard-étale ([GW2] Theorem 18.42): For $f\colon X \to Y$ locally of finite presentation and $x \in X$, $y = f(x)$, f is étale at x if and only if there exist affine open neighborhoods $U \subseteq X$ of x and $V = \operatorname{Spec} R \subseteq Y$ of y where $U \cong \operatorname{Spec}(R[T]/(f))_g$ with $f, g \in R[T]$ and f' a unit in the localization $R[T]_g$.
(4) If $f\colon X \to Y$ is smooth at $x \in X$, then there exists an open neighborhood U of x such that $f_{|U}$ can be factorized as $U \to \mathbb{A}^n_Y \to Y$ with $U \to \mathbb{A}^n_Y$ étale.

3. Serre duality

General references: [GW2] Ch. 25 and the references given there; [AK], [H] III.7.

We now come back to Serre duality. The explicit computation of the cohomology groups $H^i(\mathbb{P}^n_R, \mathscr{O}(d))$, R some ring, that we have done in *Algebraic Geometry 2* shows that for every line bundle $\mathscr{L}$ on $X := \mathbb{P}^n_R$ we have a perfect pairing

$$H^i(X, \mathscr{L}) \times H^{n-i}(X, \mathscr{L}^\vee \otimes \omega_X) \to H^n(X, \omega_X) = R.$$

Here $\omega_X := \mathscr{O}(-n-1) \cong \bigwedge^n \Omega_{X/R}$. In particular, we obtain isomorphisms $H^{n-1}(X, \mathscr{L}^\vee \otimes \omega_X) \cong H^i(X, \mathscr{L})^\vee$.

The goal of this section is to understand how this generalizes. We will (mostly) content ourselves with understanding the situation for a proper (or even projective) k-scheme X (where k is some field).

(3.1) The abstract approach.

Using the machinery of derived categories and a suitable version of the Brown representability theorem for triangulated categories, Neeman has proved that for a morphism $f\colon X \to S$ of noetherian (or more generally: qcqs) schemes, the derived pushforward functor $Rf_*\colon D_{\mathrm{qcoh}}(X) \to D_{\mathrm{qcoh}}(S)$ admits a right adjoint $f^\times$. For $X \to \operatorname{Spec}(k)$ proper, we then call $\omega_X^\bullet := f^\times \mathscr{O}_{\operatorname{Spec} k} \in D_{\mathrm{qcoh}}(X)$ the *dualizing complex* of X. Since $f^\times$ is by definition right adjoint to Rf_*, for every $F \in D_{\mathrm{qcoh}}(X)$ (and in particular for every quasi-coherent $\mathscr{O}_X$-module F) we obtain the following very general form of Grothendieck-Serre duality,

$$H^i(X; F)^\vee = \operatorname{Hom}_{D(k)}(Rf_*F[i], k) = \operatorname{Hom}_{D(X)}(F[i], \omega_X^\bullet) = \operatorname{Ext}^{-i}_{\mathscr{O}_X}(F, \omega_X^\bullet).$$

The formula simplifies for example if F is a locally free $\mathscr{O}_X$-module (because then $\operatorname{Ext}^{-i}_{\mathscr{O}_X}(F, \omega_X^\bullet) = H^{-i}(X, F^\vee \otimes_{\mathscr{O}_X} \omega_X^\bullet))$ and especially if the complex $\omega_X^\bullet$ is concentrated in a single degree. This is the case if X is smooth over k, in which case $\omega_X^\bullet = \left(\bigwedge^{\dim X} \Omega_{X/k}\right)[\dim X]$.

Now consider a closed immersion $i\colon X \to Y$ of S-schemes (where we again assume that all schemes are noetherian). See [GW2] Section (25.8). To describe the functor $i^\times$, we start with the following elementary result.

Lemma 3.1. *Let $\varphi\colon A \to B$ be a ring homomorphism, let M be an A-module and let N be a B-module. Then $\operatorname{Hom}_A(B, M)$ is a B-module in a natural way, which we denote by $\operatorname{Hom}_A^B(B, M)$. We have identifications*

$$\operatorname{Hom}_A(N, M) = \operatorname{Hom}_B(N, \operatorname{Hom}_A^B(B, M))$$

functorial in M and N. (Here on the left we consider N as an A-module via φ.

Globalizing this, for a closed immersion $i\colon Z \to X$ of schemes and an $\mathscr{O}_X$-module $\mathscr{F}$, we write $\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}_Z, \mathscr{F}) := i^* \mathscr{H}om_{\mathscr{O}_X}(i_*\mathscr{O}_Z, \mathscr{F})$.

Lemma 3.2. *Let $i\colon Z \to X$ be a closed immersion of schemes. The functor $\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}, -)$ from the category of $\mathscr{O}_X$-modules to the category of $\mathscr{O}_Z$-modules is right adjoint to the direct image functor i_*.*

The lemma "formally" implies an analogous adjunction between the derived functors $R\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}, -)$ and Li_*. Since i_* is exact, we can identify $Li_* = i_* = Ri_*$. One checks that $i_*R\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}_Z, \mathscr{F}) = R\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_Z, \mathscr{F})$, i.e., when considered as an $\mathscr{O}_X$-module, this is just the usual $R\mathscr{H}om$ functor.

Note however that we do not immediately get a description of $i^\times$ because $i^\times$ is the right adjoint of $Ri_*\colon D_{\mathrm{qcoh}}(Z) \to D_{\mathrm{qcoh}}(X)$, and in general an object of the form $R\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}_Z, \mathscr{F})$ will not lie in $D_{\mathrm{qcoh}}(Z)$ (cf. Example 3.13 below). This is however true under an additional assumption, namely for F in $D^+_{\mathrm{qcoh}}(X)$ (or in $D^+_{\mathrm{coh}}(X)$) the complex $R\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}_Z, F)$ lies in $D^+_{\mathrm{qcoh}}(Z)$ (or in $D^+_{\mathrm{coh}}(Z)$, respectively). Therefore we have

Proposition 3.3. *Let $i\colon Z \to X$ be a closed immersion of noetherian schemes. Let F in $D^+_{\mathrm{qcoh}}(X)$. Then $i^\times F = R\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}_Z, F)$.*

This gives us a strategy of constructing a dualizing complex/"dualizing sheaf" for closed subschemes of projective space. Formally, we will not use any of the results above; they only serve as a motivation/explanation of why the definitions below are sensible.

(3.2) The dualizing sheaf of a projective scheme 1.

We start by slightly generalizing the statement of Serre duality for projective space, as follows. Nov. 22, 2023

Proposition 3.4. *Let k be a field, $n \geq 1$, $X = \mathbb{P}^n_k$. Write $\omega_X = \bigwedge^n \Omega_{X/k} \cong \mathscr{O}_X(-n-1)$.*

(1) *We have $H^n(X, \omega_X) \cong k$ (and we fix one such isomorphism).*

(2) *For every coherent $\mathscr{O}_X$-module $\mathscr{F}$, the natural pairing*

$$\mathrm{Hom}(\mathscr{F}, \omega_X) \times H^n(X, \mathscr{F}) \to H^n(X, \omega_X) \cong k$$

is perfect, i.e., it induces an isomorphism $\mathrm{Hom}(\mathscr{F}, \omega_X) \cong H^n(X, \mathscr{F})^\vee$.

(3) *For every $i \geq 0$, we have an isomorphism*

$$\mathrm{Ext}^i(\mathscr{F}, \omega_X) \cong H^{n-i}(X, \mathscr{F})^\vee.$$

(See below for a brief reminder on the Ext functor.)

Proof. We have already seen Part (1), as well as Part (2) for $\mathscr{F}$ a line bundle. Clearly, then (2) also holds for finite direct sums of line bundles. For a general $\mathscr{F}$, we can find a presentation

$$\mathscr{E}' \to \mathscr{E} \to \mathscr{F} \to 0$$

where $\mathscr{E}$ and $\mathscr{E}$ are sums of line bundles and the sequence is exact. Since the functors $\mathrm{Hom}(-, \omega_X)$ and $H^n(X, -)^\vee$ are left exact and the result holds for $\mathscr{E}$ and $\mathscr{E}'$, it follows for $\mathscr{F}$.

In view of Part (2), Part (3) follows if we can show that the δ-functors $(\mathrm{Ext}^i(-, \omega_X))_i$ and $(H^{n-i}(X, -)^\vee))_i$ are universal. To prove this, it is enough to show they are coeffaceable. But any $\mathscr{F}$ can be written as a quotient of an $\mathscr{O}_X$-module of the form $\mathscr{O}_X(-d)^{\oplus N}$ with $d \gg 0$, and both functors vanish on such sheaves for $i > 0$. (In fact, $d > n + 1$ is enough since then $\omega_X(d)$ has no higher cohomology.) □

We use Parts (1) and (2) of the previous proposition to define the notion of *dualizing sheaf* for an arbitrary proper k-scheme X. (This already characterizes a dualizing sheaf, and in fact we will see below that Part (3) will not hold in general; this is related to the fact that the dualizing sheaf captures only one cohomology object of the dualizing complex of the previous section, so unless that complex is concentrated in a single degree, the dualizing sheaf will not capture the full duality.)

Definition 3.5. *Let k be a field and let X be a proper k-scheme of dimension n. A coherent $\mathscr{O}_X$-module ω_X such that there exist isomorphisms*

$$\mathrm{Hom}(\mathscr{F}, \omega_X) = H^n(X, \mathscr{F})^\vee,$$

functorial on $\mathscr{F}$, is called a dualizing sheaf *on X. In other words, a dualizing sheaf is a coherent $\mathscr{O}_X$-module ω_X, together with a homomorphism $H^n(X, \omega_X) \to k$ (the element of $H^n(X, \omega_X)^\vee$ corresponding to* id_{ω_X}*, called the* trace map*), if for every coherent $\mathscr{O}_X$-module the natural pairing*

$$\mathrm{Hom}(\mathscr{F}, \omega_X) \times H^n(X, \mathscr{F}) \to H^n(X, \omega_X) \xrightarrow{t} k$$

is perfect, i.e., induces an isomorphism $\mathrm{Hom}(\mathscr{F}, \omega_X) \cong H^n(X, \omega_X)^\vee$.

Since a dualizing sheaf is defined as the object representing a certain functor, it is clear that it is unique up to unique isomorphism (if it exists).

By the above, $\bigwedge^n \Omega_{\mathbb{P}^n_k/k}$ is a dualizing sheaf on projective space $\mathbb{P}^n_k$. We will see below that more generally for every *smooth* projective k-scheme X of dimension n, $\bigwedge^n \Omega_{X/k}$ is a dualizing sheaf on X.

Theorem 3.6. *For every proper scheme over a field, a dualizing sheaf exists.*

We will not prove the theorem here, but will rather concentrate on the case of projective schemes. (In terms of the abstract theory, writing $\omega_X^\bullet$ for the dualizing complex of X defined above, one can show that $H^i(\omega_X^\bullet) = 0$ for all $i \notin [-\dim(X), 0]$; it follows that $H^{-\dim(X)}(\omega_X^\bullet)$ is a dualizing sheaf on X.)

(3.3) $\mathscr{E}xt$ sheaves.

By and large we follow [H] III.6. See also [GW2] Sections (F.52), (21.18), (21.21), (22.17) for more general results in the context of derived categories.

Definition 3.7.

(1) *Let $\mathcal{A}$ be an abelian category with enough injectives. We define the* Ext *functor from $\mathcal{A}$ to the category of abelian groups (for $\mathscr{F}$ in $\mathcal{A}$) as* $\operatorname{Ext}^i(\mathscr{F}, -) = R^i \operatorname{Hom}(\mathscr{F}, -)$. *(One can show that*
$$\operatorname{Ext}^i(\mathscr{F}, \mathscr{G}) = \operatorname{Hom}_{D(\mathcal{A})}(\mathscr{F}, \mathscr{G}[i]).$$
This identity can be used as a definition of Ext *groups for arbitrary abelian categories.)*

(2) *Now let X be a ringed space. For an $\mathscr{O}_X$-module $\mathscr{F}$ we define the $\mathscr{E}xt$ sheaf functor from ($\mathscr{O}_X$-Mod) to ($\mathscr{O}_X$-Mod) by $\mathscr{E}xt^i(\mathscr{F}, -) = R^i \mathscr{H}om(\mathscr{F}, -)$.*

More explicitly, the definition means that we can compute $\operatorname{Ext}^i(\mathscr{F}, \mathscr{G})$ using an injective resolution of $\mathscr{G}$.

Proposition 3.8. *Let X be a ringed space and let $\mathscr{F}$, $\mathscr{G}$ be $\mathscr{O}_X$-modules.*
$$\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_{|U} = \mathscr{E}xt^i_{\mathscr{O}_U}(\mathscr{F}_{|U}, \mathscr{G}_{|U}).$$

Proof. This follows from the following fact about injective $\mathscr{O}_X$-modules: If $\mathscr{I}$ is an injective $\mathscr{O}_X$-module, then the restriction $\mathscr{I}_{|U}$ is an injective $\mathscr{O}_U$-module. (In fact, the restriction functor admits an exact left adjoint functor, the extension by zero functor, and hence preserves the class of injective objects.) □

Since $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X, -)$ is the identity functor, we have $\mathscr{E}xt^i(\mathscr{O}_X, \mathscr{G}) = \mathscr{G}$ if $i = 0$, and $= 0$ if $i > 0$. Similarly, we have $\operatorname{Ext}^i(\mathscr{O}_X, \mathscr{G}) = H^i(X, \mathscr{G})$ (where we use that H^i which we defined as the derived functor of $\Gamma\colon \mathrm{Ab}_X \to \mathrm{Ab}$ restricts to the derived functor of $\Gamma\colon (\mathscr{O}_X\text{-Mod} \to \mathrm{Ab})$. Nov. 27, 2023

Remark 3.9.

(1) Since the functor $\operatorname{Hom}(-, \mathscr{I})$ for an injective $\mathscr{O}_X$-module $\mathscr{I}$ is exact, for an $\mathscr{O}_X$-module $\mathscr{G}$ the family $(\operatorname{Ext}^i(-, \mathscr{G}))_i$ is a δ-functor (in particular, we obtain a long exact cohomology sequence when we plug in a short exact sequence in the first entry). A similar remark holds for $\mathscr{E}xt$ sheaves.

(2) Similarly, one shows that $\operatorname{Ext}^i(\mathscr{F}, \mathscr{G})$ and $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ can be computed using a projective resolution of $\mathscr{F}$. This is useful working in the derived category of modules over a ring. But when working with $\mathscr{O}_X$-modules for a ringed space X (even a noetherian scheme) in general projective objects are very rare. Note that even the structure sheaf $\mathscr{O}_X$ is not a projective $\mathscr{O}_X$-module in general.

(3) The previous two remarks show that an object $\mathscr{F}$ of an abelian category $\mathcal{A}$ with enough projectives is projective if and only if $\operatorname{Ext}^1(\mathscr{F}, \mathscr{G}) = 0$

for all $\mathscr{G}$. More generally, $\mathscr{F}$ has projective dimension n (i.e., admits a projective resolution of length n, but not of any smaller length) if and only if $\mathrm{Ext}^{n+1}(\mathscr{F},\mathscr{G}) = 0$ for all $\mathscr{G}$. (Take a projective resolution $\mathscr{P}_\bullet \to \mathscr{F} \to 0$, then $\cdots \to \mathscr{P}_i \to \mathrm{Im}(\mathscr{P}_i \to \mathscr{P}_{i-1}) := \mathscr{I}_i \to 0$ is a projective resolution of $\mathscr{I}_i$ and $\mathrm{Ext}^{n+1-i}(\mathscr{I}_i,\mathscr{G}) \cong \mathrm{Ext}^{n+1}(\mathscr{F},\mathscr{G})$. Now do induction.)

Therefore it is useful to study whether other types of resolutions (specifically, a resolution by locally free $\mathscr{O}_X$-modules of finite rank) can be used for computing $\mathscr{E}xt$ sheaves.

Proposition 3.10. *The $\mathscr{E}xt$ sheaves can be computed using a locally free finite rank resolution for the first entry, i.e.,*

$$\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = H^i(\mathscr{H}om(\mathscr{E}_\bullet,\mathscr{G})),$$

if $\cdots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$ *is exact and all* $\mathscr{E}_i$ *are locally free of finite rank.*

Proof. Both sides are δ-functors in $\mathscr{G}$, and they agree for $i = 0$. Both sides vanish for $i > 0$ and $\mathscr{G}$ injective (for the left hand side, use that $\mathscr{H}om(-,\mathscr{G})$ is exact since injectivity is preserved under restriction to opens, cf. the proof of Proposition 3.8) and hence are universal δ-functors. □

Question. Do you see why for the Ext groups this cannot possibly hold? Asked differently, what goes wrong in the proof, if we replace $\mathscr{E}xt$ by Ext and $\mathscr{H}om$ by Hom? (*Hint.* Where have we used that the $\mathscr{E}_i$ are locally free?)

Corollary 3.11.
(1) *Let A be a noetherian ring and let $X = \mathrm{Spec}\, A$. Let M, N be A-modules. Assume that M is finitely generated. For every $i \geq 0$ we have*

$$\mathscr{E}xt^i_{\mathscr{O}_X}(\widetilde{M},\widetilde{N}) \cong \mathrm{Ext}^i_A(M,N)^\sim.$$

(2) *Let X be a noetherian scheme and let $\mathscr{F}$, $\mathscr{G}$ be coherent $\mathscr{O}_X$-modules. Then for all $i \geq 0$ the $\mathscr{O}_X$-module $\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ is coherent.*

Proof. For Part (1), let $P_\bullet \to M \to 0$ be a resolution of M by finite projective A-modules. We can use this resolution to compute $\mathrm{Ext}^i_A(M,N)$, and can use $\widetilde{P}_\bullet$ to compute the left hand side. The result follows since $\mathscr{H}om_{\mathscr{O}_X}(\widetilde{P},\widetilde{N}) = \mathrm{Hom}_A(P,N)^\sim$ (because P is of finite presentation). Part (2) follows from Part (1) since for N finitely generated, $\mathrm{Ext}^i_A(M,N)$ is finitely generated, as the considerations in the proof of (1) show. □

It is also true, and can be shown using the local-to-global spectral sequence for Ext (see below) and the vanishing of cohomology of quasi-coherent modules on affine schemes in positive degrees, that in Part (1) of the Corollary we have $\Gamma(X, \mathscr{E}xt^i_{\mathscr{O}_X}(\widetilde{M},\widetilde{N})) = \mathrm{Ext}_{\mathscr{O}_X}(\widetilde{M},\widetilde{N})$, but it does not seem easier/easy to directly identify that Ext group (which has to be computed in the category of all $\mathscr{O}_X$-modules) with $\mathrm{Ext}^i_A(M,N)$ (which we can view as Ext *of $\widetilde{M}$ and*

$\widetilde{N}$ in the category of all quasi-coherent $\mathscr{O}_X$-modules). I did not explain this well in class.

Corollary 3.12. *Let X be a noetherian scheme, let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules and assume that $\mathscr{F}$ is coherent. Then for all $x \in X$ we have*

$$\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})_x = \mathrm{Ext}^i_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x).$$

Proof. By Proposition 3.8 we can reduce to the case that $X = \operatorname{Spec} A$ is affine. Choose a resolution $\mathscr{P}_\bullet \to \mathscr{F} \to 0$ of $\mathscr{F}$ by free $\mathscr{O}_X$-modules $\mathscr{P}_i$. Passing to the stalks at x, we obtain a free resolution of $\mathscr{F}_x$ as an $\mathscr{O}_{X,x}$-module.

By the above, $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = \mathscr{H}om(\mathscr{P}_i, \mathscr{G})$, and $\mathscr{H}om(\mathscr{P}_i, \mathscr{G})_x = \mathrm{Hom}(\mathscr{P}_{i,x}, \mathscr{G}_x)$ since $\mathscr{P}_i$ is of finite presentation (this is enough since the result clearly holds for $\mathscr{O}_X$ in the first entry, thus for finite free modules, and by the five lemma follows for modules of finite presentation). □

Example 3.13. Let R be a discrete valuation ring with field of fractions K, $X = \operatorname{Spec} R$, $\eta \in X$ the generic point. Then

$$\mathscr{H}om_{\mathscr{O}_X}(\bigoplus_{\mathbb{N}} R, R)_\eta = \left(\prod_{\mathbb{N}} R\right) \otimes_R K \neq \prod_{\mathbb{N}} K = \mathrm{Hom}_K(K \otimes_R \bigoplus_{\mathbb{N}} R, K).$$

Proposition 3.14. *Let X be a ringed space, let $\mathscr{F}$, $\mathscr{G}$ be $\mathscr{O}_X$-modules, and $\mathscr{E}$ a locally free $\mathscr{O}_X$-modules. We write $\mathscr{E}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X)$. Then for all $i \geq 0$ we have natural identifications*

$$\mathrm{Ext}^i(\mathscr{F} \otimes \mathscr{E}, \mathscr{G}) = \mathrm{Ext}^i(\mathscr{F}, \mathscr{G} \otimes \mathscr{E}^\vee)$$

and

$$\mathscr{E}xt^i(\mathscr{F} \otimes \mathscr{E}, \mathscr{G}) = \mathscr{E}xt^i(\mathscr{F}, \mathscr{G} \otimes \mathscr{E}^\vee) = \mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) \otimes \mathscr{E}^\vee.$$

Proof. All the above functors are δ-functors in $\mathscr{G}$ which are moreover effaceable and hence universal. Therefore it is enough to show the equalities in the case $i = 0$ where they are clear. □

Excursion: Spectral sequences 3.15. See [GW2] Sections (F.21) – (F.25), (F.50). Other references are [We], [Gr], [EGA] III_1 §11.

Nov. 29, 2023

Proposition 3.16. (Leray spectral sequence)

(1) *Let $f\colon X \to Y$ be a morphism of ringed spaces and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. There is a convergent spectral sequence*

$$E_2^{pq} = H^p(Y, R^q f_* \mathscr{F}) \Longrightarrow H^{p+q}(X, \mathscr{F}).$$

(2) *Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms of ringed spaces. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. There is a convergent spectral sequence*

$$E_2^{pq} = R^p g_*(R^q f_* \mathscr{F}) \Longrightarrow R^{p+q}(g \circ f)_* \mathscr{F}.$$

Proof. These are the Grothendieck spectral sequences for the compositions $\Gamma(Y,-)\circ f_* = \Gamma(X,-)$ and $g_*\circ f_* = (g\circ f)_*$. In fact, the hypotheses are satisfied because for every injective (and even every flasque) $\mathscr{O}_X$-module $\mathscr{I}$, the direct image $f_*\mathscr{I}$ is flasque and hence acyclic for $\Gamma(Y,-)$ and g_*, resp. □

Proposition 3.17. (Čech vs. derived functor cohomology) *Let X be a ringed space and let $\mathscr{U}$ be an open cover of X. There is a convergent spectral sequence*

$$E_2^{pq} = \check{H}^p(\mathscr{U}, \mathscr{H}^q(\mathscr{F})) \Longrightarrow H^{p+q}(X,\mathscr{F}).$$

Here $\mathscr{H}^q(\mathscr{F})$ denotes the presheaf $U\mapsto H^q(U,\mathscr{F})$.

Proof. This is the Grothendieck spectral sequence for the composition of functors $\check{H}^0$ from the category of presheaves(!) on X to the category of abelian groups, and the inclusion ι of the category of $\mathscr{O}_X$-modules into the category of presheaves of $\mathscr{O}_X$-modules. Note that ι preserves the property of being injective because it has an exact left adjoint functor (namely sheafification). The composition is just the global section functor on the category of sheaves. One concludes by noting that $\mathscr{H}^q$ is the q-th right derived functor of ι, and that $\check{H}^p(\mathscr{U},-)$ is the p-th right derived functor of $\check{H}^0(\mathscr{U},-)$ (on the category of presheaves of $\mathscr{O}_X$-modules. For the latter fact, the key is to show that for $\mathscr{I}$ an injective presheaf we have $\check{H}^p(\mathscr{U},\mathscr{I})=0$ for all $p>0$. This follows by defining a sheaf version of the Čech complex and showing that it is exact in positive degrees; see [GW2] Lemma 21.76 and Lemma 21.77 and/or [H] Ch. III, Proposition 4.3. □

Proposition 3.18.

(1) (The local-to-global spectral sequence for Ext) *Let X be a ringed space and let $\mathscr{F}$, $\mathscr{G}$ be $\mathscr{O}_X$-modules. We have a convergent spectral sequence*

$$E_2^{pq} = H^p(X, \mathscr{E}xt^q(\mathscr{F},\mathscr{G})) \Longrightarrow \mathrm{Ext}^{p+q}(\mathscr{F},\mathscr{G}).$$

(2) *Let $i\colon Z\to X$ be a closed immersion of schemes (or of arbitrary ringed spaces), let $\mathscr{F}$ be an $\mathscr{O}_Z$-module and $\mathscr{G}$ an $\mathscr{O}_X$-module. We have a convergent spectral sequence*

$$E_2^{pq} = \mathrm{Ext}^p_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{O}_Z,\mathscr{G})) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(i_*\mathscr{F},\mathscr{G}).$$

Proof. (1). This is the Grothendieck spectral sequence for the composition $\Gamma(X,-)\circ \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},-) = \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F},-)$. Note that for $\mathscr{I}$ injective, the sheaf $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{I})$ is flasque (given $j\colon U\to X$ an open subscheme, and a homomorphism $\mathscr{F}_{|U}\to\mathscr{I}_{|U}$, we get $j_!(\mathscr{F}_{|U})\to\mathscr{I}$, and from the injectivity of $\mathscr{I}$ and the injection $j_!(\mathscr{F}_{|U})\to\mathscr{F}$ a map $\mathscr{F}\to\mathscr{I}$ extending the map given on U).

(2) Again, this is a Grothendieck spectral sequence, now for the composition $\mathrm{Hom}_{\mathscr{O}_Z}(\mathscr{F},-)\circ \mathscr{H}om^{\mathscr{O}_Z}_{\mathscr{O}_X}(\mathscr{O}_Z,-) = \mathrm{Hom}_{\mathscr{O}_X}(i_*\mathscr{F},-)$. Recall that we have checked this equality in Section 3.1, where we expressed it by saing that

$\mathscr{H}om^{\mathscr{O}_Z}_{\mathscr{O}_X}(\mathscr{O}_Z, -)$ is the right adjoint of i_*. (Strictly speaking, we should write $\mathscr{E}xt^{\mathscr{O}_Z,q}_{\mathscr{O}_X}(\mathscr{O}_Z, \mathscr{G})$, but we omit the upper $\mathscr{O}_Z$ at this point.) To obtain the Grothendieck spectral sequence in this situation, we need to check that for $\mathscr{I}$ injective, $\mathscr{H}om^{\mathscr{O}_Z}_{\mathscr{O}_X}(\mathscr{O}_Z, \mathscr{I})$ is injective. In fact, by the adjunction, $\mathrm{Hom}(-, \mathscr{H}om^{\mathscr{O}_Z}_{\mathscr{O}_X}(\mathscr{O}_Z, \mathscr{I})) = \mathrm{Hom}(i_*-, \mathscr{I})$ which is exact. □

Corollary 3.19. *Let k be a field, $i\colon X \to \mathbb{P}^n_k$ a closed immersion and $\mathscr{O}_X(1) := i^*\mathscr{O}_{\mathbb{P}^n_k}(1)$. Let $\mathscr{F}, \mathscr{G}$ be coherent $\mathscr{O}_X$-modules. Then for all $n \ge 0$ there exists $d_0 \ge 0$ (depending on $\mathscr{F}, \mathscr{G}$ and p) such that for all $d \ge d_0$ we have*

$$\mathrm{Ext}^n_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(d)) = \Gamma(X, \mathscr{E}xt^n_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(d))).$$

Proof. Consider the spectral sequence of Proposition 3.18 (1). For fixed n, only finitely many terms of the E_2 term play a role in computing the values E_r^{pq} with $r \ge 2$, $p + q = n$ (note that the values vanish anyway unless $p, q \ge 0$). We choose d_0 so that for all $d \ge d_0$, all $0 \le q < n$ and all $p > 0$, the terms $H^p(X, \mathscr{E}xt^q(\mathscr{F}, \mathscr{G})(d))$ vanish. (Such a d_0 exists for each q, since $\mathscr{E}xt^q(\mathscr{F}, \mathscr{G})$ is a coherent $\mathscr{O}_X$-module, see *Algebraic Geometry 2*, and the maximum of all these gives the desired bound.) But fixing such a d, the E_2 terms for all p, q with $p + q = n$ are equal to the E_∞ terms and are all zero except when $p = 0$, $q = n$. It follows that the limit term $\mathrm{Ext}^n_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(d))$ equals $E_2^{0,n} = \Gamma(X, \mathscr{E}xt^n_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(d)))$. □

(3.4) The dualizing sheaf of a projective scheme 2.

With the results on $\mathscr{E}xt$ sheaves, we can construct a dualizing sheaf for projective schemes over a field. Dec. 4, 2023

We need the following lemma. See below (Corollary 3.31) for a different proof of the lemma.

Lemma 3.20. *Let k be a field, $n \ge 1$, and let $X \subseteq \mathbb{P}^n_k$ be a closed subscheme, $r = n - \dim X$. Let $\omega = \bigwedge^n \Omega_{\mathbb{P}^n_k/k}$. Then $\mathscr{E}xt^i_{\mathscr{O}_{\mathbb{P}^n_k}}(\mathscr{O}_X, \omega) = 0$ for all $i < r$.*

Proof. Fix i and write $\mathscr{F} = \mathscr{E}xt^i_{\mathscr{O}_{\mathbb{P}^n_k}}(\mathscr{O}_X, \omega)$. Since $\mathscr{F}$ is coherent, it is enough to show that $H^0(\mathbb{P}^n_k, \mathscr{F}(d)) = 0$ for all $d \gg 0$. By Corollary 3.19 and the duality theorem for $\mathbb{P}^n_k$, we have

$$H^0(\mathbb{P}^n_k, \mathscr{F}(d)) \cong \mathrm{Ext}^i_{\mathscr{O}_{\mathbb{P}^n_k}}(\mathscr{O}_X(-d), \omega) \cong H^{n-i}(\mathbb{P}^n_k, \mathscr{O}_X(-d)).$$

We can compute the cohomology group on the right as a cohomology group on X. This has to vanish in degrees $> \dim X$ by the Grothendieck vanishing theorem. But $n - i > \dim X$ is equivalent to $i < r$. □

Theorem 3.21. *Let k be a field, $n \geq 1$ and let $i\colon X \to \mathbb{P}^n_k$ be a closed immersion. Let $r = n - \dim X$ be the codimension of X. Let $\omega = \omega_{\mathbb{P}^n_k}$ Then $\omega_X = \mathscr{E}xt^r_{\mathscr{O}_{\mathbb{P}^n_k}}(\mathscr{O}_X, \omega)$ is a dualizing sheaf on X.*

Proof. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. We need to show that

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E}xt^r(\mathscr{O}_X, \omega)) \cong H^{\dim X}(X, \mathscr{F})^\vee.$$

We use the spectral sequence of Proposition 3.18 (2) to evaluate the term on the left. By the lemma, the E_2 term of bidegree $(0, r)$ equals the E_∞ term, and all other terms with bidegree (p, q) with $p + q = r$ vanish, so that we obtain

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E}xt^r(\mathscr{O}_X, \omega)) \cong \mathrm{Ext}^r(\mathscr{F}, \omega).$$

By duality on $\mathbb{P}^n$, the right hand term is isomorphic to $H^{n-r}(\mathbb{P}^n, \mathscr{F})^\vee = H^{\dim(X)}(X, \mathscr{F})^\vee$. The identification is functorial in $\mathscr{F}$, and thus the theorem is proved. $\square$

(3.5) Cohen-Macaulay schemes.

References: [GW2] Ch. 19; [M2] Ch. 16, 17, [BH] 1.1, 1.2

Definition 3.22. *Let A be a ring and let M be an A-module.*

(1) *A sequence $f_1, \dots, f_r \in A$ is called a* regular sequence *for M (or an M-*regular sequence*) if*
   (a) *for all i the element f_i is $M/(f_1, \dots, f_{i-1})$-regular, i.e., multiplication by f_i defines an injective endomorphism of $M/(f_1, \dots, f_{i-1})$ and*
   (b) *$M/(f_1, \dots, f_r) \neq 0$.*

(2) *Let I be an ideal of A. The* depth *of M with respect to I is the maximum length of an M-regular sequence of elements contained in I. We denote it by* $\mathrm{depth}_A(I, M)$*. If I is the maximal ideal of a local ring A, we write* $\mathrm{depth}_A(M) := \mathrm{depth}_A(I, M)$*, and we write* $\mathrm{depth}(A) = \mathrm{depth}_A(A)$*.*

We start with the following lemma. In the proof of the lemma (and also afterwards) we use the notion of *associated prime ideal* of a module. See [M2] §6. (Over non-noetherian rings, the notion of weakly associated prime ideal is sometimes more useful, see [Stacks] 0546.)

Lemma 3.23. *Let A be a ring, $I \subseteq A$ an ideal, M an A-module. The ideal I contains an M-regular element if and only if* $\mathrm{Hom}_A(A/I, M) = 0$*.*

Proof. If I contains an M-regular element x, then multiplication by x on $\mathrm{Hom}_A(A/I, M)$ at the same time is the zero map (since $x \in I$), and injective (since x is M-regular). Conversely, assume that all elements of I annihilate some element of M. This is equivalent to saying that I is contained in the union of all associated prime ideals of M (i.e., all prime ideals of A of the form $\mathrm{ann}(m) := \{x \in A;\ xm = 0\}$). By prime avoidance, I is contained in one of them, say $I \subseteq \mathfrak{p} = \mathrm{ann}(m)$. Then $x \to xm$ induces an injection $A/\mathfrak{p} \to M$, hence a non-zero map $A/I \to M$. $\square$

Proposition 3.24. *Let A be a noetherian ring, $I \subseteq A$ an ideal and M be a finitely generated A-module.*

(1) *We have*

$$\operatorname{depth}_A(I, M) \le \dim(M) := \dim(\operatorname{Supp}(M)).$$

(2) *We have*

$$\operatorname{depth}_A(I, M) = \min\{i;\ \operatorname{Ext}^i_A(A/I, M) \neq 0\}.$$

(3) *Every maximal M-regular sequence in I (i.e., a regular sequence that cannot be further extended) consists of* $\operatorname{depth}_A(I, M)$ *elements.*

(4) *If A is local, then the property of a sequence $f_1, \dots, f_r$ of being regular is independent of the order of the f_i. (But not for general noetherian rings A.)*

Indications of some of the proofs. (1). The minimal prime ideals of $\operatorname{Supp}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A);\ M_\mathfrak{p} \neq 0\}$ (a closed subset of $\operatorname{Spec}(A)$) are associated prime ideals of M. We now do induction on $\operatorname{depth}_A(I, M)$. If $\operatorname{depth}_A(I, M) = 0$, then there is nothing to do. Otherwise, there exists an M-regular element $x \in I$, so x does not lie in any of the minimal prime ideals in $\operatorname{Supp}(M)$. Thus $0 \le \dim(M/xM) < \dim(M)$. By induction, if $x_1, \dots, x_n$ is a regular sequence for M, then $\dim(M) \ge n$.

(2), (3). Let M be an A-module, and let $x_1, \dots, x_n \in I$ be an M-regular sequence. We prove by induction on n that $\operatorname{depth}_A(I, M) > n$ if and only if the given sequence can be extended to an M-regular sequence of length $n+1$ in I, if and only if $\operatorname{Ext}^m_A(A/I, M) = 0$ for all $m \le n$. The case $n = 0$ is clear by the above lemma. For $n > 0$ the short exact sequence

$$0 \to M \xrightarrow{m \mapsto x_1 m} M \to M/x_1M \to 0$$

induces an exact sequence

$$\operatorname{Ext}^{n-1}_A(A/I, M) \to \operatorname{Ext}^{n-1}_A(A/I, M/x_1M) \to \operatorname{Ext}^n_A(A/I, M) \xrightarrow{x_1 \cdot -} \operatorname{Ext}^n_A(A/I, M)$$

By induction, $\operatorname{Ext}^{n-1}(A/I, M) = 0$. Since $x_1 \in I$, the map on the right of the above sequence also is $= 0$. That means that we have an isomorphism

$$\operatorname{Ext}^{n-1}_A(A/I, M/x_1M) \cong \operatorname{Ext}^n_A(A/I, M).$$

By induction, we obtain a chain of isomorphisms

$$\operatorname{Ext}^n_A(A/I, M) \cong \operatorname{Ext}^{n-1}_A(A/I, M/x_1M) \cong \dots \cong \operatorname{Hom}_A(A/I, M/(x_1, \dots, x_n)M).$$

The claim now follows from the previous lemma.

We skip the proof of (4) for the moment; it follows from the connection with the Koszul complex – see below or see any of the references given above. $\square$

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For an A-module M we denote by $\operatorname{projdim}_A(M)$ the projective dimension of M, i.e., the minimal length of a resolution $0 \to P_\ell \to \dots \to P_0 \to M \to 0$ of M by projective A-modules.

Theorem 3.25. (Auslander-Buchsbaum formula, [M2] Theorem 19.1) *If A is a local noetherian ring and M a finitely generated A-module of finite projective dimension, then*

$$\mathrm{depth}(A) = \mathrm{projdim}_A(M) + \mathrm{depth}_A(M).$$

Fact 3.26.

(1) A local noetherian ring A is regular if and only if every finitely generated A-module has finite projective dimension ([M2] Theorem 19.2). More precisely, for A regular local, the projective dimension of finite A-modules is bounded by $\dim A$. (Below we will only use the latter statement, which is the easier direction of the result. The key point is to show that for any local noetherian ring R, the projective dimension of any finite R-module is at most the projective dimension of the residue class field, considered as an R-module. This statement follows without much work from a version of the local criterion of flatness, see e.g. [AK] III.5. If R is regular, then the residue class field has finite projective dimension, since the Koszul complex (see below) is a projective resolution of length $\dim(R)$.)

(2) Since the formation of Ext for finite modules over a noetherian ring commutes with localization, it follows that every finite module over a regular noetherian ring of finite dimension (but not necessarily local) has finite projective dimension.

Definition 3.27.

(1) *A local noetherian ring A is called a* Cohen-Macaulay ring*, if* $\mathrm{depth}(A) = \dim(A)$.

(2) *A noetherian ring A is called a* Cohen-Macaulay ring*, if for every $\mathfrak{p} \in \mathrm{Spec}(A)$ the localization $A_\mathfrak{p}$ is Cohen-Macaulay.*

(3) *A locally noetherian scheme X is called a* Cohen-Macaulay scheme*, if for every $x \in X$ the local ring $\mathscr{O}_{X,x}$ is Cohen-Macaulay.*

One can show that localizations of Cohen-Macaulay rings are again Cohen-Macaulay. In particular in Part (2) of the definition it is enough to check localizations with respect to maximal ideals.

Example 3.28.

(1) Every 0-dimensional noetherian ring is Cohen-Macaulay.

(2) Let A be a 1-dimensional noetherian ring. Then A is Cohen-Macaulay if and only if A does not have any "embedded associated prime ideals" (i.e., non-minimal associated prime ideals). In particular, if A is reduced, then A is Cohen-Macaulay. (If A is reduced, then every associated prime ideal is minimal. Indeed, assume there are prime ideals $\mathfrak{p} \subsetneq \mathfrak{q} = \mathrm{ann}(x)$. Then $x\mathfrak{q} = 0 \subseteq \mathfrak{p}$ and hence $x \in \mathfrak{p} \subset \mathfrak{q}$, but then $x^2 = 0$.)

(3) For any $d \geq 2$, there exist noetherian rings of dimension d which are not Cohen-Macaulay. For example, for k a field, neither $k[X,Y,Z]/(XY,XZ)$ (not equidimensional, cf. below), nor $k[W,X,Y,Z]/(WY,WZ,XY,XZ)$ are Cohen-Macaulay.

Facts on Cohen-Macaulay rings 3.29.
(1) Every regular noetherian ring is Cohen-Macaulay. (It is not too hard to see that every minimal generating system of the maximal ideal of a regular local ring is a regular sequence.)
(2) Every connected Cohen-Macaulay scheme locally of finite type over a field is equi-dimensional, i.e., all its irreducible components have the same dimension.
(3) If A is a Cohen-Macaulay ring, then A does not have embedded associated prime ideals (i.e., all associated prime ideals of the A-module A are minimal prime ideals of A). In particular, if X is a generically reduced Cohen-Macaulay scheme, then X is reduced.
(4) If A is a local Cohen-Macaulay ring with maximal ideal $\mathfrak{m}$, then $f_1, \dots, f_r \in \mathfrak{m}$ form an A-regular sequence if and only if $\dim A/(f_1, \dots, f_r) = \dim(A) - r$. In this case, the quotient $A/(f_1, \dots, f_r)$ is again Cohen-Macaulay.

We close the section by a local proof of Lemma 3.20. *This was skipped in the class.*

Proposition 3.30. *Let A be a ring, $I \subseteq A$ an ideal, M an A-module. Then we have*

$$\mathrm{depth}_A(I, M) = \inf\{\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p});\ \mathfrak{p} \in V(I)\}.$$

Proof. An M-regular sequence in I maps to an $M_\mathfrak{p}$-regular sequence under the natural homomorphism $A \to A_\mathfrak{p}$ for any prime ideal $\mathfrak{p} \in V(I)$, so we have $\leq$. To show the other inequality, we can replace M by the quotient $M/(x_1, \dots, x_n)M$ where $x_\bullet$ is a maximal M-regular sequence in I, und thus suppose that the left hand side is $= 0$. We then need to show that there exists $\mathfrak{p} \in V(I)$ with $\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$. As in the proof of Lemma 3.23, the assumption implies that I is contained in some associated prime ideal $\mathfrak{p}$, and we obtain an injective A-module homomorphism $A/\mathfrak{p} \to M$. Tensoring with $A_\mathfrak{p}$ preserves the injectivity, so we see that $\mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \neq 0$, and Lemma 3.23 gives the result. $\square$

Corollary 3.31. *Let k be a field, $n \geq 1$, and let $X \subseteq \mathbb{P}^n_k$ be a closed subscheme, $r = n - \dim X$. Let $\mathscr{E}$ be a locally free $\mathscr{O}_{\mathbb{P}^n_k/k}$-module of finite rank. Then $\mathscr{E}xt^i_{\mathscr{O}_{\mathbb{P}^n_k}}(\mathscr{O}_X, \mathscr{E}) = 0$ for all $i < r$.*

Proof. It is enough to prove that all stalks of this (coherent) sheaf at closed points vanish, so it is enough to show that for every closed point $x \in \mathbb{P}^n_k$ and $i < r$, writing $A := \mathscr{O}_{\mathbb{P}^n_k, x}$,

$$\mathrm{Ext}^i_A(\mathscr{O}_{X,x}, \mathscr{E}_x) = 0.$$

Denoting by $I \subseteq A$ the ideal corresponding to the closed subscheme X, and noting that $\mathscr{E}$ is a locally free sheaf of finite rank, so that $\mathscr{E}_x$ is a finite free A-module, this means precisely that we need to show $\mathrm{depth}_A(I, A) \geq r$. By Proposition 3.30, it is enough to show that $\mathrm{depth}(A_\mathfrak{p}) \geq r$ for all $\mathfrak{p} \in V(I)$.

But A is regular (it is the localization of the polynomial ring in n variables over k at some maximal ideal), so $\operatorname{depth}(A_{\mathfrak{p}}) = \dim(A_{\mathfrak{p}}) = n - \dim(A/\mathfrak{p})$, and the claim follows. □

(3.6) The Koszul complex.

References: [GW2] Ch. 19; [M2] Ch. 16, [BH] 1.6

Let A be a ring. Let $e_1, \dots, e_r$ denote the standard basis of the free A-module A^r. For $f_\bullet = (f_1, \dots, f_r)$ with $f_i \in A$ we define the *Koszul complex* as the (chain) complex $K_\bullet(f_\bullet)$

$$0 \to \bigwedge^r A^r \to \bigwedge^{r-1} A^r \to \cdots \to \bigwedge^2 A^r \to \bigwedge^1 A^r \to \bigwedge^0 A^r \to 0$$

(with $\bigwedge^p A^r$ in degree p, i.e., the differential decreases the degree) with differentials

$$\bigwedge^p A^r \to \bigwedge^{p-1} A^r, \quad e_{i_1} \wedge \cdots \wedge e_{i_p} \mapsto \sum_{\mu=1}^{p} (-1)^{\mu+1} f_\mu e_{i_1} \wedge \cdots \wedge \widehat{e_{i_\mu}} \wedge \cdots \wedge e_{i_p}.$$

One checks that this in fact defines a complex. For an A-module M we write $K_\bullet(f_\bullet, M) := K_\bullet(f_\bullet) \otimes_A M$ and denote by $H_p(f_\bullet, M)$ the homology groups.

Proposition 3.32. *Let A be a ring, M an A-module and $f_1, \dots, f_r \in A$.*
1. *We have $H_0(f_\bullet, M) = M/(f_1, \dots, f_r)M$.*
2. *If $f_1, \dots, f_r$ is an M-regular sequence, then $H_p(f_\bullet, M) = 0$ for all $p > 0$.*
3. *If A is a noetherian local ring with maximal ideal $\mathfrak{m}$, $M \neq 0$ is finitely generated, $f_1, \dots, f_r \in \mathfrak{m}$ and $H_1(f_\bullet, M) = 0$, then $f_1, \dots, f_r$ is an M-regular sequence.*

Proof. Part (1) follows directly from the definition. For Parts (2) and (3), we check the cases $r = 1, 2$. See the references for the general case, e.g., [M2] Theorem 16.5. For $r = 1$ the assertion is clear immediately. For $r = 1$, the Koszul complex, tensored with M, is the complex

$$0 \to M \xrightarrow{x \mapsto (-f_2 x, f_1 x)} M^2 \xrightarrow{(x,y) \mapsto f_1 x + f_2 x} M \to 0.$$

Assume that f_1, f_2 is an M-regular sequence. Then clearly the map $M \to M^2$ is injective. Let $x, y \in M$ with $f_1 x + f_2 y = 0$, so y is mapped to zero under $M/(f_1) \xrightarrow{f_2 \cdot} M/(f_1)$ and hence $y \in f_1 M$, say $y = f_1 y'$. We find that $f_1(x + f_2 y') = 0$, so $x = -f_2 y'$. Thus $(x, y) = (-f_2 y', f_1 y') \in \operatorname{Im}(M \to M^2)$.

Conversely, assume A is local, M is finitely generated and that H_1 of the above complex vanishes. It is then straighforward to check that $M/f_1 M \xrightarrow{f_2} M/f_1 M$ is injective. To prove that multiplication by f_1 is an injection $M \to M$, by the Lemma of Nakayama it is enough to show that multiplication by f_2 on the (finitely generated, since A is noetherian) A-module $\operatorname{Ker}(f_{1|M})$ is surjective. But if $f_1 x = 0$, then $(x, 0) \in A^2$ maps to zero and hence must have the form $(-f_2x', f_1x')$ for some $x' \in A$, i.e., $x' \in \mathrm{Ker}(f_{1|M})$ and $x = -f_2x'$. □

(3.7) The dualizing sheaf of a projective scheme 3.

We can now prove that projective Cohen-Macaulay schemes "satisfy Serre duality in all degrees", and that this property characterizes them. In terms of the dualizing complex, connected Cohen-Macaulay schemes X are characterized by the property that the dualizing complex is concentrated in a single degree (namely $-\dim X$). Since "full duality" always holds for the dualizing complex, from this point of view it is clear that for connected Cohen-Macaulay schemes it holds for the dualizing sheaf. Dec. 11,

We start with the following strengthening, for finitely generated modules over noetherian rings, of Remark 3.9. 2023

Lemma 3.33. *Let A be a noetherian ring and let M be a finitely generated A-module.*

(1) *If $\mathrm{Ext}^1_A(M, N) = 0$ for all finitely generated A-modules N, then M is projective.*
(2) *If $\mathrm{Ext}^{d+1}_A(M, N) = 0$ for all finitely generated A-modules N, then* $\mathrm{projdim}(M) \le d$.
(3) *Now assume that A is a regular ring and that* $\dim A < \infty$. *If* $\mathrm{Ext}^i_A(M, A) = 0$ *for all $i > d$, then* $\mathrm{projdim}(M) \le d$.

Proof. (1). Since M is finitely generated, there is a surjection $A^r \to M$. Let N be its kernel, a finitely generated A-module since A is noetherian. From the short exact sequence $0 \to N \to A^r \to M \to 0$, we obtain a long exact sequence

$$\mathrm{Hom}_A(M, A^r) \to \mathrm{Hom}_A(M, M) \to \mathrm{Ext}^1_A(M, N) = 0$$

which gives us a splitting of our short exact sequence.

(2). There exists an exact sequence

$$0 \to R \to P_{d-1} \to \cdots \to P_0 \to M \to 0$$

with all P_i projective of finite rank and R a finitely generated A-module. We obtain that $\mathrm{Ext}^1(R, N) = \mathrm{Ext}^{d+1}(M, N) = 0$ for all finitely generated N, whence R is projective.

(3). We show that $\mathrm{Ext}^i_A(M, N) = 0$ for all $i > d$ and all finite A-modules N by descending induction on i. The induction start holds since M has finite projective dimension (here we use that A is regular, Fact 3.26). For the induction step, consider a finite A-module N and a short exact sequence $0 \to K \to A^r \to N \to 0$. We obtain an exact sequence

$$\mathrm{Ext}^i_A(M, A^r) \to \mathrm{Ext}^i_A(M, N) \to \mathrm{Ext}^{i+1}_A(M, K).$$

For $i > d$, the term on the left is 0 by assumption. By induction, the term on the right is 0. Thus the term in the middle vanishes, too. □

Theorem 3.34. *Let k be a field, and let X be a connected projective k-scheme of dimension n. Let $\mathscr{O}_X(1)$ denote the pullback of the corresponding twisting sheaf under some closed embedding of X into a projective space over k. Let ω_X denote the dualizing sheaf of X.*

(1) *For all coherent $\mathscr{O}_X$-modules $\mathscr{F}$ and all $i \geq 0$, there are k-vector space homomorphisms, functorial in $\mathscr{F}$,*

$$\mathrm{Ext}^i(\mathscr{F}, \omega) \to H^{n-i}(X, \mathscr{F})^\vee.$$

(2) *The following are equivalent:*
   (i) *The scheme X is Cohen-Macaulay.*
   (ii) *For every locally free $\mathscr{O}_X$-module $\mathscr{E}$ of finite rank, every $i < n$ and $d \gg 0$, we have*
   $$H^i(X, \mathscr{E}(-d)) = 0.$$
   (iii) *The morphisms $\mathrm{Ext}^i(\mathscr{F}, \omega) \to H^{n-i}(X, \mathscr{F})^\vee$ of Part (1) are isomorphisms for all i and all coherent $\mathscr{O}_X$-modules $\mathscr{F}$.*

Proof. (1). The functors on each side are δ-functors, and we have the homomorphism (even an isomorphism) for $i = 0$ by the definition of a dualizing sheaf. It is therefore sufficient to show that the left hand side is a *universal* δ-functor, which we show by proving that it is coeffaceable. In fact, given $\mathscr{F}$, we can write it as a quotient of some $\mathscr{O}_X(-d)^{\oplus N}$, and $\mathrm{Ext}^i(\mathscr{O}_X(-d), \omega) = H^i(X, \omega(d))$ vanishes for $i > 0$ and $d \gg 0$.

(2). Clearly, (iii) implies (ii) (cf. the following corollary).

To show (ii) $\Rightarrow$ (iii), in view of Part (1) it is enough that the functor $\mathscr{F} \mapsto H^{N-i}(X, \mathscr{F})^\vee$ is a universal δ-functor. But (ii) shows that it is coeffaceable.

To finish the proof, we show the equivalence of (i) and (ii). Recall that we have embedded $X \subseteq \mathbb{P}^N_k$ in order to define $\mathscr{O}_X(1)$ (or, expressed using the terminology of very ample line bundles: fixing the "very ample" line bundle $\mathscr{O}_X(1)$ on X defines an embedding into projective space).

Note that for every locally free $\mathscr{O}_X$-module $\mathscr{E}$, every i and $d \gg 0$ we have, using duality on $\mathbb{P}^N_k$ and Corollary 3.19,

$$\begin{aligned} H^i(X, \mathscr{E}(-d)) &= H^i(\mathbb{P}^N_k, \mathscr{E}(-d)) = \mathrm{Ext}^{N-i}_{\mathscr{O}_{\mathbb{P}^N_k}}(\mathscr{E}, \omega_{\mathbb{P}^N_k}(d)) \\ &= \Gamma(\mathbb{P}^N_k, \mathscr{E}xt^{N-i}(\mathscr{E}, \omega_{\mathbb{P}^N_k}(d))). \end{aligned} \tag{3.7.1}$$

In particular, it follows that $H^i(X, \mathscr{E}(-d)) = 0$ for all $d \gg 0$ if and only if $\mathscr{E}xt^{N-i}(\mathscr{E}, \omega_{\mathbb{P}^N_k}(d)) = 0$ for all $d \gg 0$.

(i) $\Rightarrow$ (ii). Let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module, and let $x \in X$ be a closed point. We can consider $\mathscr{E}_x$ as an $\mathscr{O}_{X,x}$-module, but also as an $\mathscr{O}_{\mathbb{P}^N_k,x}$-module, and over either of these rings it has the same depth. Now the depth over $\mathscr{O}_{X,x}$ is $\dim \mathscr{O}_{X,x} = \dim X = n$, since X by assumption is Cohen-Macaulay and $\mathscr{E}_x$ is free over $\mathscr{O}_{X,x}$. On the other hand, $\mathscr{O}_{\mathbb{P}^N_k,x}$ is regular so that the depth of $\mathscr{E}$, by the Auslander-Buchsbaum formula, equals $N - \mathrm{projdim}_{\mathscr{O}_{\mathbb{P}^N_k,x}}(\mathscr{E}_x)$. We find that $\mathrm{projdim}_{\mathscr{O}_{\mathbb{P}^N_k,x}}(\mathscr{E}_x)$ equals the codimension of X inside $\mathbb{P}^N_k$.

But then $\mathscr{E}xt^i_{\mathscr{O}_{\mathbb{P}^N_k}}(\mathscr{E}, \mathscr{G})_x = \mathrm{Ext}(\mathscr{E}_x, \mathscr{G}_x) = 0$ for all $i > N - n$ and all $\mathscr{G}$. We apply this to $\mathscr{G} = \omega_{\mathbb{P}^N_k}(d)$ for $d \gg 0$ and get, by (3.7.1), that $H^i(X, \mathscr{E}(-d)) = 0$ for $i < n$, as desired.

(ii) $\Rightarrow$ (i). Conversely, applying (ii) to $\mathscr{E} = \mathscr{O}_X$, we obtain from (3.7.1) that $\mathscr{E}xt^{N-i}(\mathscr{O}_X, \omega_{\mathbb{P}^N_k})(d) = 0$ for all $d \gg 0$ and all $i < n$. Thus all stalks of these sheaves vanish. But the stalk of the second entry is $\cong \mathscr{O}_{\mathbb{P}^N_k} =: A$, so we get $\mathrm{Ext}^i_A(A/I, A) = 0$ for all $i > N - n$, where I denotes the stalk at x of the ideal sheaf defining X. This implies $\mathrm{projdim}_A(A/I) \leq N - n$ (cf. Lemma 3.33), and via the Auslander-Buchsbaum formula, that $\mathrm{depth}(A/I) \geq n$. But $A/I = \mathscr{O}_{X,x}$ has dimension n, so it follows that $\mathscr{O}_{X,x}$ is Cohen-Macaulay. □

Corollary 3.35. *Let k be a field and let X by a connected Cohen-Macaulay projective k-scheme of dimension n. For every locally free $\mathscr{O}_X$-module $\mathscr{E}$ and all i, we have isomorphisms*

$$H^i(X, \mathscr{E}^\vee \otimes \omega) \cong H^{n-i}(X, \mathscr{E})^\vee.$$

(3.8) The dualizing sheaf of a regularly embedded projective scheme.

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Even for a projective Cohen-Macaulay scheme X, it is not so easy to describe the dualizing sheaf "explicitly", and it will not in general be a locally free $\mathscr{O}_X$-module. For a slightly smaller class of (projective) schemes, the situation is much better, however, as we will discuss now.

Definition 3.36. *A closed immersion $i\colon Z \to X$ of noetherian schemes with corresponding ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_X$ is called* regular *if every $z \in Z$ admits an affine open neighborhood $U \subseteq X$ such that the ideal $\Gamma(U, \mathscr{I}) \subset \Gamma(U, \mathscr{O}_X)$ is generated by a regular sequence.*

In the situation of the definition, the $\mathscr{O}_Z$-module $\mathscr{I}/\mathscr{I}^2$ is locally free (locally, a regular sequence generating the ideal induces a basis). See [M2] Theorem 16.2 or [BouA] Ch. X §9 no. 7 Thm. 1. Conversely, if X is regular and $\mathscr{I}/\mathscr{I}^2$ is locally free, then the closed immersion is regular, see [M2] Theorem 19.9.

From Remark 3.29 (4) we obtain the following result.

Proposition 3.37. *Let $i\colon Z \to X$ be a regular closed immersion. If X is Cohen-Macaulay, then Z is Cohen-Macaulay.*

Proposition 3.38. *Let S be a noetherian scheme and let Z, X be smooth S-schemes of finite type. Let $i\colon Z \to X$ be a closed immersion. Then i is regular.*

Proof. See [GW2] Theorem 19.30. The key fact from commutative algebra is the following: For a local noetherian ring A and an ideal I such that the quotient A/I is regular, the ideal I is generated by a regular sequence if (and only if) A is regular. See [BouAC] Ch. X §2 no. 2, Cor. de Prop. 2.

If $S = \operatorname{Spec} k$ is a field (or more generally, whenever X is a regular schemes), we can also argue as follows. Let $\mathscr{I}$ denote the ideal sheaf attached to i. By Proposition 2.42 and Proposition 2.50 we know that $\mathscr{I}/\mathscr{I}^2$ is a locally free $\mathscr{O}_Z$-module. Therefore i is regular by the remark following the definition of regular closed immersion. □

More generally one can show that for a k-scheme X the property of admitting a regular immersion into a smooth k-scheme is a property of the local rings $\mathscr{O}_{X,x}$ (being *lci rings*, where lci stands for *locally complete intersection*). So if X admits a regular closed embedding into $\mathbb{P}^n_k$ (or any smooth k-scheme) every closed immersion of X into a projective space (or in fact into any smooth k-scheme) is regular. See [GW2] Corollary 19.44, Proposition 19.49, Proposition 19.50.

Theorem 3.39. (cf. [GW2] Theorem 25.57) *Let $i\colon Z \to X$ be a regular closed immersion of noetherian schemes, let $r = \dim X - \dim Z$, and let $\mathscr{I} \subset \mathscr{O}_X$ be the corresponding quasi-coherent ideal sheaf. Then for every coherent $\mathscr{O}_X$-module $\mathscr{F}$ there is an isomorphism*

$$\mathscr{E}xt^r_{\mathscr{O}_X}(\mathscr{O}_Z, \mathscr{F}) \cong \mathscr{H}om_{\mathscr{O}_Z}(\bigwedge^r \mathscr{I}/\mathscr{I}^2, i^*\mathscr{F}).$$

Proof. We first consider the local situation, i.e., we fix $x \in X$ and consider an affine open neighborhood $\operatorname{Spec} A$ of x, where Z is given by an ideal $I \subseteq A$ that can be generated by a regular sequence $f_1, \dots, f_r$. Let $K_\bullet$ denote the Koszul complex for this sequence. In view of Proposition 3.32, our assumptions ensure that (after shrinking the neighborhood further, if necessary) $K_\bullet$ is a projective resolution of A/I. So we can use it to compute the Ext groups we are interested in. Let M be an A-module. Then we have

$$\begin{aligned}
\operatorname{Ext}^r_A(A/I, M) &= H^r(\operatorname{Hom}_A(K_\bullet, M)) \\
&= M/IM \\
&= \operatorname{Hom}_{A/I}(A/I, M/IM) \\
&= \operatorname{Hom}_{A/I}(\bigwedge^r(I/I^2), M/IM).
\end{aligned}$$

The second equality is an easy computation. (In fact, the Koszul complex is “self-dual” (shifting appropriately), [GW2] Section (22.4).)

The final identification of this chain is obtained from the isomorphism

$$\bigwedge^r(I/I^2) \to A/I, \quad f_1 \wedge \dots \wedge f_r \mapsto 1.$$

In view of Corollary 3.11, this gives the result in the affine case.

To conclude, we check that the local isomorphisms are independent of the choice of regular sequence generating I. It is then clear that they glue to give the desired natural isomorphism. Given another regular sequence $g_1, \dots, g_r$ generating I, we can write $g_j = \sum_i a_{ij} f_i$ for elements $a_{ij} \in A$ and the residue classes of the g_i will be another basis of I/I^2. We then have $g_1 \wedge \dots \wedge g_r = \det((a_{ij})_{i,j}) f_1 \wedge \dots \wedge f_r$.

On the other hand, one checks that the map $A^r \to A^r$ given by the matrix $(a_{ij})_{i,j}$ induces a quasi-isomorphism between the Koszul complexes for the two regular sequences. The induced automorphism of M/IM is multiplication by $\det((a_{ji})_{i,j}) = \det((a_{ij})_{i,j})$. □

Corollary 3.40. *Let k be a field, X a smooth projective k-scheme of dimension n, and ω_X its dualizing sheaf. Then*

$$\omega_X \cong \bigwedge^n \Omega_{X/k}.$$

Proof. Fix a closed immersion $i\colon X \to \mathbb{P}^N_k$, denote by $\mathscr{I}$ the corresponding ideal sheaf, and let $\omega_{\mathbb{P}^N_k}$ be the dualizing sheaf of $\mathbb{P}^N_k$. By Proposition 3.38, i is a regular immersion. Thus by Theorem 3.21 and Theorem 3.39, we have

$$\begin{aligned}\omega_X &\cong \mathscr{E}xt^{N-n}_{\mathscr{O}_{\mathbb{P}^N_k}}(\mathscr{O}_X, \omega_{\mathbb{P}^N_k}) \cong \mathscr{H}om_{\mathscr{O}_X}(\bigwedge^{N-n} \mathscr{I}/\mathscr{I}^2, \omega_{\mathbb{P}^N_k|X}) \\ &\cong \left(\bigwedge^{N-n} \mathscr{I}/\mathscr{I}^2\right)^\vee \otimes_{\mathscr{O}_X} \omega_{\mathbb{P}^N_k|X}.\end{aligned}$$

By Proposition 2.42 the right hand side is isomorphic to $\bigwedge^n \Omega_{X/k}$. □

(3.9) Higher direct images.

Let $f\colon X \to Y$ be a morphism of ringed spaces. We denote by $R^i f_*$ the right derived functors of the direct image functor f_* from the category of abelian sheaves on X to the category of abelian sheaves on Y, and call the $R^i f_*$ the *higher direct image* (or *derived direct image*) functors. Dec. 20, 2023

As a special case, if Y is a point, then f_* can be identified with the global section functor, and then $R^i f_*$ is the cohomology functor $H^i(X, -)$.

Proposition 3.41. *Let $f\colon X \to Y$ be a morphism of ringed spaces, and let $\mathscr{F}$ be an abelian sheaf on X. Then $R^i f_* \mathscr{F}$ is the sheafification of the presheaf*

$$V \mapsto H^i(f^{-1}(V), \mathscr{F})$$

on Y.

Proof. Consider the commutative diagram

$$\begin{tikzcd} (\mathrm{Ab}_X) \arrow[r, "\iota"] \arrow[d] & (\mathrm{PreAb}_X) \arrow[d, "f_*^p"] \\ (\mathrm{Ab}_Y) & (\mathrm{PreAb}_Y) \arrow[l, "-^\sharp"] \end{tikzcd}$$

where in the right hand column we have the categories of presheaves of abelian groups on X and Y, respectively, f_*^p is the "presheaf direct image" functor (defined in the same way as f_* for sheaves), and where ι is the inclusion and $-^\sharp$ is the sheafification functor.

Since f_*^p and sheafification are exact functors, the Grothendieck spectral sequence for the composition of functors shows that $R^i f_*\mathscr{F}$ is the sheafification of the presheaf $f_*^p R^i\iota\mathscr{F}$. Computing this presheaf (using an injective resolution of $\mathscr{F}$, and using that restriction to open subsets preserves the property of being injective) we obtain the result. □

Corollary 3.42. *Let $f\colon X \to Y$ be a morphism of ringed spaces, let $i \geq 0$ and let $\mathscr{F}$ be an abelian sheaf on X. For any open subspace $V \subseteq Y$ we have $(R^i f_*\mathscr{F})_{|V} = R^i f_{|f^{-1}(V),*}(\mathscr{F}_{|f^{-1}(V)})$.*

Lemma 3.43. *Let $f\colon X \to Y$ be a morphism of ringed spaces, let $i \geq 0$ and let $\mathscr{F}$ be an abelian sheaf on X. Then flasque sheaves are acyclic for f_*, i.e., for any flasque abelian sheaf $\mathscr{F}$ on X and $i > 0$ we have $R^i f_*\mathscr{F} = 0$.*

Proof. This follows from Proposition 3.41, the fact that flasque sheaves are acyclic for the global section functor and that the restriction of a flasque sheaf to an open is again flasque. □

Similar as for cohomology groups, the previous lemma together with the fact that injective sheaves are flasque implies that for $\mathscr{O}_X$-modules we can compute the higher direct images in the category of $\mathscr{O}_Y$-modules.

Proposition 3.44. *Denote by F the direct image functor $(\mathscr{O}_X\text{-Mod}) \to (\mathscr{O}_Y\text{-Mod})$. For every $\mathscr{O}_X$-module $\mathscr{F}$ and every $i \geq 0$, we have $R^i f_*\mathscr{F} = R^i F\mathscr{F}$ (as abelian sheaves).*

As a preparation for showing that the higher direct image functor for qcqs morphisms preserves quasi-coherence, we need the following lemma. The case $i = 0$ is a characterizing property of quasi-coherence. The general case can be proved using tools that we have at hand, but it requires several steps; we skip the proof here.

Lemma 3.45. ([GW2] Lemma 22.26) *Let X be a qcqs scheme and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Let $t \in \Gamma(X, \mathscr{O}_X)$, write $X_t = \{x \in X;\ t(x) \neq 0\}$, the open subscheme of X where t does not vanish, and fix $i \geq 0$. Then there is an isomorphism $H^i(X, \mathscr{F})_t \cong H^i(X_t, \mathscr{F}_{|X_t})$, and these isomorphisms are functorial in $\mathscr{F}$.*

Proposition 3.46. *Let $f\colon X \to Y$ be a qcqs morphism of schemes. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Let $i \in \mathbb{Z}_{\geq 0}$.*
(1) *Assume that Y is affine. Then*

$$R^i f_*\mathscr{F} \cong H^i(X, \mathscr{F})^\sim.$$

(2) *The $\mathscr{O}_Y$-module $R^i f_*\mathscr{F}$ is quasi-coherent.*

Proof. (1). We use the previous lemma. We denote by $\mathscr{H}^i$ the presheaf $U \mapsto H^i(U, \mathscr{F})$ on X, so that $R^i f_* \mathscr{F}$ is the sheafification of $f_* \mathscr{H}^i$, as we have already seen. Let $\varphi\colon A \to \Gamma(X, \mathscr{O}_X)$ be the ring homomorphism obtained from f. For $s \in A$ we obtain

$$\Gamma(D(s), f_* \mathscr{H}^i) = H^i(X_{\varphi(s)}, \mathscr{F}) = H^i(X, \mathscr{F})_s = \Gamma(D(s), H^i(X, \mathscr{F})^\sim).$$

One checks that these maps are compatible with restriction along inclusions $D(t) \subseteq D(s)$. Thus the presheaf $f_* \mathscr{H}^i$ and the sheaf $H^i(X, \mathscr{F})^\sim$ agree on the basis of the topology given by principal open subsets, and that implies the claim.

If X is noetherian, one can alternatively argue as follows. The equality holds for $i = 0$, so it is enough to show that both sides are universal δ-functors *from the category of quasi-coherent $\mathscr{O}_X$-modules to the category of quasi-coherent $\mathscr{O}_Y$-modules*. (It is important to restrict to quasi-coherent modules here since obviously the claim does not hold in degree 0 in general.) For the left hand side, this is clear. The right hand side is a δ-functor, since this is true for the cohomology on X and since the $\sim$-operation is exact.

Now since X is noetherian, every quasi-coherent $\mathscr{O}_X$-module can be embedded into a flasque quasi-coherent $\mathscr{O}_X$-module (see [H] Ch. III Cor. 3.6; also see [GW2] Section (22.18) for stronger results), and then it follows from Lemma 3.43 that the δ-functor on the right hand side is effaceable.

Part (2) follows from Part (1) since we can compute the higher direct images locally on Y. $\square$

(3.10) Relative versions of some previous results.

From the Leray spectral sequence one obtains the following result. We have already proved Part (2) for closed immersions f in *Algebraic Geometry 2*.

Corollary 3.47. *Let $f\colon X \to Y$ be an affine morphism of schemes (i.e., for every affine open $V \subseteq Y$ the inverse image $f^{-1}(V)$, an open subscheme of X, is affine). Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module and let $i \geq 0$,*

(1) *We have $R^i f_* \mathscr{F} = 0$ for all $i > 0$.*

(2) *We have natural isomorphisms*

$$H^p(Y, f_* \mathscr{F}) \cong H^p(X, \mathscr{F}).$$

(3) *For every morphism $g\colon Y \to Z$ of schemes we have functorial isomorphisms*

$$R^i (f \circ g)_* \mathscr{F} \cong R^i g_* (f_* \mathscr{F}).$$

Proof. Part (1) follows directly from the vanishing of cohomology of quasi-coherent modules on affine schemes in positive degrees. Parts (2) and (3) then follow from the Leray spectral sequences (Proposition 3.16). $\square$

The finiteness results for cohomology of coherent modules on projective schemes have the following relative version.

Proposition 3.48. *Let $f\colon X \to Y$ be a projective morphism of noetherian schemes, i.e., there exists a closed immersion $i\colon X \to \mathbb{P}^N_Y$ of Y-schemes, and let $\mathscr{O}_X(1) := i^*\mathscr{O}_{\mathbb{P}^N_Y}(1)$. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module.*

(1) *For d sufficiently large, the natural homomorphism $f^*f_*\mathscr{F}(d) \to \mathscr{F}(d)$ is surjective.*
(2) *For all $i \geq 0$, the higher direct image $R^if_*\mathscr{F}$ is coherent.*
(3) *For $i > 0$ and d sufficiently large, $R^if_*\mathscr{F}(n) = 0$.*

Finally, we state a relative version of duality. See [Kl], Theorem 20, Theorem 21, Corollary 25, for an elementary (i.e., not using derived categories) proof and several more general/more precise results, including a version for coherent, not necessarily locally free, modules; see [GW2] Ch. 25, e.g., Prop. 25.26, for a more general result in terms of the dualizing complex.

Theorem 3.49. *Let $f\colon X \to Y$ be a flat projective morphism of noetherian schemes. Assume that all fibers of f are Cohen-Macaulay and equi-dimensional of the same dimension r.*

(1) *There exists a coherent $\mathscr{O}_X$-module ω_f (the* relative dualizing sheaf*) such that*
$$R^if_*(\omega_f \otimes_{\mathscr{O}_X} \mathscr{E}^\vee) \cong (R^{r-i}f_*\mathscr{E})^\vee$$
for every $i = 0, \dots, r$ and every locally free $\mathscr{O}_X$-module $\mathscr{E}$ of finite rank (and such that ω_f gives rise to a similar, but slightly more difficult to state duality for arbitrary coherent $\mathscr{O}_X$-modules (not necessarily locally free), which also characterizes ω_f up to unique isomorphism).
(2) *If X and Y are smooth and of finite type over some noetherian base scheme S of relative dimensions d_X and d_Y, respectively, then*
$$\omega_f = \bigwedge^{d_X} \Omega_{X/S} \otimes f^*\left(\bigwedge^{d_Y} \Omega_{Y/S}\right)^\vee.$$

4. Cohomology and base change

General references: [GW2] Ch. 23, in particular Sections (23.24)–(23.28), [H] Ch. III.12, [Mu] §5.

(4.1) The base change homomorphism.

Jan. 8, 2024

We consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

where f is quasi-compact and quasi-separated.

For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and $i \geq 0$ we have a natural homomorphism of $\mathscr{O}_{S'}$-modules, called the *base change homomorphism*,

$$g^*R^if_*\mathscr{F} \longrightarrow R^if'_*(g')^*\mathscr{F},$$

and these are functorial in $\mathscr{F}$. We are interested in finding conditions when these homomorphisms are isomorphisms.

For $i = 0$, the natural homomorphism $g^*f_* \to f'_*(g')^*$ of functors corresponds, by adjunction, to a morphism $f_* \to g_*f'_*(g')^* = f_*g'_*(g')^*$, which we obtain by applying f_* to the unit $\mathrm{id} \to g'_*(g')^*$ of the adjunction between $(g')^*$ and g'_*. To handle the case of $i > 0$, we use the following remark. (Cf. [GW2] Section (23.28) for a construction in the framework of derived categories.)

Remark 4.1. (Construction of the base change homomorphism) We may work locally on S and S' (and in the general case, use gluing). If $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$ are affine, then in view of Proposition 3.46 we need to construct a natural map

$$H^i(X, \mathscr{F}) \otimes_R R' \to H^i(X', (g')^*\mathscr{F}) = H^i(X, \mathscr{F} \otimes_R R'),$$

where the final identification uses Corollary 3.47 (2). Since cohomology is a functor, the natural R-module homomorphism $\mathscr{F} \to \mathscr{F} \otimes_R R'$, $f \mapsto f \otimes 1$, induces a homomorphism $H^i(X, \mathscr{F}) \to H^i(X, \mathscr{F} \otimes_R R')$. Since the right hand side is an R'-module, we obtain the desired map $H^i(X, \mathscr{F}) \otimes_R R' \to H^i(X, \mathscr{F} \otimes_R R')$.

Proposition 4.2. *If in the above situation g is flat and f is separated, then for every i the base change homomorphism*

$$g^*R^if_*\mathscr{F} \longrightarrow R^if'_*(g')^*\mathscr{F},$$

is an isomorphism.

Proof. We can again work locally on S and S', say $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$ and can then compute the cohomology groups as Čech cohomology groups for a finite affine open cover $\mathscr{U}$ of X and $\mathscr{U}' := (g')^{-1}(\mathscr{U})$ of X'. (Note that v is an affine morphism since S, S' are affine, and the property of being an affine morphism is stable under base change.) But $C^\bullet(\mathscr{U}', (g')^*\mathscr{F}) = C^\bullet(\mathscr{U}, \mathscr{F}) \otimes_R R'$. We obtain a homomorphism $C^\bullet(\mathscr{U}, \mathscr{F}) \to C^\bullet(\mathscr{U}', (g')^*\mathscr{F})$ of complexes, and this induces the desired base change homomorphisms.

Now g is flat, so $- \otimes_R R'$ commutes with passing to the cohomology objects, and the proposition follows. Instead of assuming that f be separated it is enough that f is qcqs. See [GW2] Section(23.28). □

Often, the base change homomorphism is most interesting, however, if g is not flat. A typical example is when g is the natural morphism $\operatorname{Spec} \kappa(s) \to S$ for a point $s \in S$.

(4.2) Proper base change for coherent modules.

From now on we will focus on the situation

(S) $S = \operatorname{Spec}(R)$ is noetherian, f proper, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module which is flat over S.

For $s \in S$, we write $X_s := X \times_S \operatorname{Spec} \kappa(s)$ for the fiber, and $\mathscr{F}_s := \mathscr{F}_{|X_s}$ for the restriction of $\mathscr{F}$ to X_s (i.e., the pullback of $\mathscr{F}$ along the projection morphism $X_s \to X$).

For the results below, we need to use that in situation (S) for the coherent $\mathscr{O}_X$-module $\mathscr{F}$ the higher direct images $R^if_*\mathscr{F}$ are again coherent for all i. This is true (for S noetherian and f proper and any coherent $\mathscr{O}_X$-module $\mathscr{F}$), but in *Algebraic Geometry 2* we proved it only for f projective. See [GW2] Ch. 23 for a proof in the general case.

Theorem 4.3. ([GW2] Theorem 23.139, Theorem 23.140)
Consider situation (S).

(1) *The Euler characteristic $s \mapsto \chi_{\mathscr{F}}(s) := \sum_i (-1)^i \dim_{\kappa(s)} H^i(X_s, \mathscr{F}_s)$ is locally constant on S.*
(2) *The map $S \to \mathbb{Z}_{\geq 0}$, $s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathscr{F}_s)$ is upper semicontinuous, i.e., for every $n \in \mathbb{Z}$ the set*

$$\{s \in S;\ \dim_{\kappa(s)} H^i(X_s, \mathscr{F}_s) \geq n\}$$

is closed in S.
(3) *Let $s \in S$. For $i \in \mathbb{Z}_{\geq 0}$, the following are equivalent.*
   (i) *The base change homomorphism $\beta^i(\kappa(s))$ for $S' = \operatorname{Spec} \kappa(s) \to S$ is surjective.*
   (ii) *After replacing S by an open neighborhood U of s, the base change homomorphism is an isomorphism for every $S' \to S$ (and the fixed sheaf $\mathscr{F}$ and degree i).*
(4) *Let $s \in S$ such that $\beta^i(\kappa(s))$ is surjective. The following are equivalent.*
   (i) *The homomorphism $\beta^{i-1}(\kappa(s))$ is surjective.*

(ii) *There exists an open neighborhood V of s such that the $\mathscr{O}_V$-module $R^i f_* \mathscr{F}_{|V}$ is finite locally free.*

In this case, the function $s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathscr{F}_s)$ is locally constant on V.

As the proof will show, in Part (4) of the theorem it is allowed to take $i = 0$, in which case condition (i) is automatically satisfied (by convention; or from a derived category point of view where we would define the functor $R^i f_* - = H^i(Rf_* -)$ which makes sense for all $i \in \mathbb{Z}$ and vanishes for $i < 0$, if applied to a complex concentrated in non-negative degrees).

All statements of the theorem are local on S, so we may assume that $S = \operatorname{Spec}(R)$ for a noetherian ring R.

In the next proposition, we write $\mathscr{F} \otimes_R M$ for the quasi-coherent $\mathscr{O}_X$-module $\mathscr{F} \otimes_{\mathscr{O}_X} f^*\widetilde{M}$. Equivalently, this is the $\mathscr{O}_X$-module with sections $\mathscr{F}(U) \otimes_R M$ on every affine open $U \subseteq X$. If $M = R'$ is an R-algebra, then we are in the situation of the base change homomorphism considered previously. A particularly important case is when $s \in \operatorname{Spec}(r)$ and $M = R' = \kappa(s)$ – then we obtain $H^i(X, \mathscr{F} \otimes \kappa(s)) = H^i(X_s, \mathscr{F}_s)$. Jan. 10, 2024

Proposition 4.4. ([GW2] Corollary 23.137, [Mu] §5, [H] III Proposition 12.2)

In the above situation, there exists a bounded complex $E^\bullet$ concentrated in degrees ≥ 0 of finitely generated projective R-modules and isomorphisms, functorial in the R-module M,

$$H^i(X, \mathscr{F} \otimes_R M) \xrightarrow{\sim} H^i(E^\bullet \otimes_R M),$$

for all $i \geq 0$.

Proof. We can compute the cohomology groups in question using Čech cohomology for a fixed finite affine open cover $\mathscr{U}$ of X, and $C^\bullet(\mathscr{U}, \mathscr{F} \otimes_R M) \cong C^\bullet(\mathscr{U}, \mathscr{F}) \otimes_R M$ since tensor product commutes with finite products. Thus the proposition follows from the following general fact in homological algebra. □

Lemma 4.5. *Let R be a noetherian ring and let $C^\bullet$ be a complex of flat R-modules such that $H^i(C^\bullet)$ is finitely generated for every i. Assume that $C^\bullet$ is bounded, concentrated in degrees $[a, b]$, say. Then there exists a bounded complex $E^\bullet$ concentrated in degrees $[a, b]$ of* finitely generated *flat R-modules and a morphism $E^\bullet \to C^\bullet$ such that for every R-module M and every i the induced homomorphism*

$$H^i(E^\bullet \otimes M) \to H^i(C^\bullet \otimes M)$$

is an isomorphism.

Proof. We start by constructing finite free R-modules E^i by descending induction on i (together with differentials $d_E^i \colon E^i \to E^{i+1}$ and maps $\psi^i \colon E^i \to C^i$) such that

(a) $d_E^{i+1} \circ d_E^i = 0, \quad d_C^i \circ \psi_i = \psi^{i+1} \circ d_E^i$,

(b) the induced map $H^{i+1}(E^\bullet) \to H^{i+1}(C^\bullet)$ is an isomorphism,
(c) the induced map $\mathrm{Ker}(d_E^i) \to H^i(C)$ is surjective.

For i large, we may set $E^i = 0$. For the induction step, to construct E^i, choose generators of the finitely generated R-module $H^i(C^\bullet)$ and lift them to elements $x_1, \dots, x_r \in \mathrm{Ker}(d_C^i)$. Also, choose generators $y_{r+1}, \dots, y_s$ of $(\psi^{i+1})^{-1}(\mathrm{Im}(d_C^i)) \cap \mathrm{Ker}(d_E^{i+1})$ and lift their images in $\mathrm{Im}(d_C^i)$ to elements $x_{r+1}, \dots, x_s \in C^i$.

Set $E^i = R^{\oplus s}$ and define ψ^i by mapping the standard basis vectors to the x_ν. Define d_E^i by mapping $x_\nu \mapsto 0$ for $\nu = 1, \dots, r$, and $x_\nu \mapsto y_\nu$ for $\nu = r+1, \dots, s$. This constructs E^i, d_E^i and ψ_i satisfying (a), (b) and (c).

One checks that this gives bounded above complex $E^\bullet$ of finite free R-modules, together with a morphism $E^\bullet \to C^\bullet$ which is a quasi-isomorphism (this is precisely property (b) above).

Next let us show that we can replace $E^\bullet$ by a bounded complex with the same properties (where we allow finite flat rather than free entries). Say $C^i = 0$ for all $i < i_0$. We replace E^i by 0 for all $i < i_0$ and replace E^{i_0} by $E^{i_0}/\mathrm{Im}(d^{i_0-1})$, so that the new complex has the same cohomology objects as the old one, and still comes with a morphism to $C^\bullet$, which we again denote by ψ. We need to show that the new E^{i_0} is flat over R. Let M_ψ be the mapping cone of $\psi\colon E^\bullet \to C^\bullet$. Since ψ is a quasi-isomorphism, the complex M_ψ is exact. Its terms have the form $E_i \oplus C_{i-1}$ and its left-most entry is E_{i_0}. We obtain an exact sequence

$$0 \to E_{i_0} \to M_{\psi, i_0+1} \to \cdots \to 0$$

where all entries, except possibly the left-most one, are flat R-modules. It follows that E_{i_0} is flat, as well. (Namely, one can inductively apply the following observation: Given a short exact sequence $0 \to M' \to M \to M'' \to 0$ with M and M'' flat, it follows by looking at the long exact Tor sequences that M' is flat, as well.)

Finally we check that the induced homomorphisms $H^i(E^\bullet \otimes_R M) \to H^i(C^\bullet \otimes_R M)$ are isomorphisms for all R-modules M. Since every R-module is a filtered direct limit of finitely generated modules, and both sides commute with direct limits, it is enough to consider finitely generated M. Write M as the quotient F/N for a finite free R-module F. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E^\bullet \otimes_R N & \longrightarrow & E^\bullet \otimes_R F & \longrightarrow & E^\bullet \otimes_R M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^\bullet \otimes_R N & \longrightarrow & C^\bullet \otimes_R F & \longrightarrow & C^\bullet \otimes_R M & \longrightarrow & 0
\end{array}$$

of complexes with exact rows (here we use that all terms of $E^\bullet$, and $C^\bullet$, respectively, are flat). This induces a similar commutative diagram of the long exact cohomology sequences. In view of (a), the middle vertical arrow induces isomorphisms on the cohomology objects. The claim now follows by descending induction on i, using suitable versions of the 5-lemma: The claim is clear for sufficiently large i since then both sides vanish. It then follows that the maps in question are surjective. Finally, invoking the 5-lemma again, it follows that they are isomorphisms.

Alternatively, for the final step one can again use the mapping cone, and use that the mapping cone for $\psi \otimes M$ is $M_\psi \otimes M$ and that this is again exact because of the flatness. □

We can now prove the first two parts of the theorem.

Proof of Theorem 4.3, Parts (1), (2). Let $E^\bullet$ be as in the lemma. Then for every $s \in S$,

$$\begin{aligned}\chi_{\mathscr{F}}(s) &= \sum_i (-1)^i \dim_{\kappa(s)} H^i(E^\bullet \otimes \kappa(s)) \\ &= \sum_i (-1)^i \dim_{\kappa(s)} E^i \otimes \kappa(s) = \sum_i (-1)^i \operatorname{rk} E^i.\end{aligned}$$

This proves Part (1).

For Part (2), write $H^i(E^\bullet \otimes \kappa(s)) = \operatorname{Coker}(E^{i-1} \otimes \kappa(s) \to \operatorname{Ker}(d^i_{E\otimes\kappa(s)}))$. Its dimension is

$$\dim(E^i \otimes \kappa(s)) - \operatorname{rk}(d^i \otimes \kappa(s)) - \operatorname{rk}(d^{i-1} \otimes \kappa(s)), \tag{4.2.1}$$

which is upper semicontinuous. □

The next two lemmas are crucial ingredients from linear algebra that go into the proof of parts (3) and (4) of the theorem. Jan. 15, 2024

Lemma 4.6. ([GW1] Proposition 8.10) *Let S be a scheme and let $\iota\colon \mathscr{U} \to \mathscr{E}$ be a homomorphism of $\mathscr{O}_S$-modules, where $\mathscr{U}$ is of finite type and $\mathscr{E}$ is locally free of finite rank. Let $s \in S$. The following are equivalent.*

(i) *In an open neighborhood of S, there exists a retraction π of ι (i.e., $\pi \circ \iota = \mathrm{id}$).*
(ii) *After replacing S with an open neighborhood of s, for every scheme morphism $f\colon T \to S$, the homomorphism $f^*\mathscr{U} \to f^*\mathscr{E}$ is injective.*
(iii) *The homomorphism $\iota \otimes \kappa(s)\colon \mathscr{U}(s) \to \mathscr{E}(s)$ between the fibers is an injective homomorphism of $\kappa(s)$-vector spaces.*
(iv) *In an open neighborhood of S, the homomorphism ι is injective and the quotient $\mathscr{E}/\mathscr{U}$ is locally free.*
(v) *After replacing S with an open neighborhood of s, the $\mathscr{O}_S$-module $\mathscr{U}$ is locally free and the dual $\iota^\vee$ is surjective.*

Proof. It is clear that (i) ⇒ (ii) ⇒ (iii), and that (iv) ⇒ (i).

(iii) ⇒ (iv). We may assume that R is local with residue class κ and that $s \in \operatorname{Spec}(R)$ is the closed point. (For the reduction to this situation, note that $\mathscr{E}/\iota(\mathscr{U})$ is of finite presentation, so if its stalk at s is free over $\mathscr{O}_{S,s}$, then $\mathscr{E}/\iota(\mathscr{U})$ is free in a neighborhood of s (*Algebraic Geometry 2*, Proposition 2.17; or [GW1] Proposition 7.27). In this case, it follows that $\iota(\mathscr{U})$ is locally free in a neighborhood of s, and hence that $\operatorname{Ker}(\iota) \subseteq \mathscr{U}$ is a direct summand. Therefore $\mathrm{Ker}(\iota)$ is of finite type in a neighborhood of s, so its support is closed. Since s is not contained in the support, it follows that the sheaf $\mathrm{Ker}(\iota)$ is zero on some neighborhood of s.)

Then it is enough to show the following: Let $i\colon N \to M$ be a homomorphism of R-modules with N finitely generated, M projective and such that the map $i\otimes\kappa\colon N\otimes\kappa \to M\otimes\kappa$ is injective. Then i is injective and $i(N)$ is a direct summand of M.

Let r_0 be a retraction of $i\otimes\kappa$. The homomorphism $M \to M\otimes\kappa \xrightarrow{r_0} N\otimes\kappa$ factors as $M \xrightarrow{r'} N \to N\otimes\kappa$ since M is projective. Then $r'\circ i$ is an endomorphism of N which induces the identity after $-\otimes\kappa$. By the lemma of Nakayama it is surjective and hence, as a surjective endomorphism of a finitely generated module, bijective ([M2] Theorem 2.4). We obtain $(r'\circ i)^{-1}\circ r'\circ i = \mathrm{id}_N$, so i admits a retraction.

(iii) $\Leftrightarrow$ (v). We already know that (iii) implies that ι is injective, and $\mathscr{U}$ is locally free around s. Thus $\iota^\vee\otimes\kappa(s) = (\iota\otimes\kappa(s))^\vee$, so the injectivity of $\iota\otimes\kappa(s)$ is equivalent to the surjectivity of $\iota^\vee\otimes\kappa(s)$, and by the lemma of Nakayama, to the surjectivity of $\iota^\vee$. □

Lemma 4.7. ([GW2] Lemma 23.122) *Let R be a ring, and let $d\colon M\to N$ be a homomorphism of finitely generated projective R-modules. Let $s\in\mathrm{Spec}(R)$. The following are equivalent.*

(i) *The natural map $\mathrm{Ker}(d)\otimes\kappa(s)\to\mathrm{Ker}(d\otimes\kappa(s))$ is surjective.*

(ii) *There exists $f\in R$ with $s\in D(f)$ such that $\mathrm{Coker}(d)\otimes_R R_f$ is a projective R_f-module.*

In this case, $\mathrm{Im}(d\otimes_R R_f)$ is a direct summand of $N\otimes_R R_f$.

Proof. (i) $\Rightarrow$ (ii). We need to show that the localization $\mathrm{Coker}(d)_\mathfrak{p}$ at the prime ideal $\mathfrak{p}$ corresponding to s is free. This means that we may assume that R is local with closed point s, and then show that $\mathrm{Coker}(d)$ is free. In this case, M and N are free, of ranks m and n, say. We write $\kappa=\kappa(s)$ and $r=\mathrm{rk}(d\otimes\kappa)$. We will show that $\mathrm{Im}(d)\subseteq N$ is a direct summand of rank r; this implies our claim.

To this end, by Lemma 4.6, it is enough to prove that $\bar d\otimes\kappa$ is injective, where $\bar d\colon M/\mathrm{Ker}(d)\to N$ is the injection induced by d. But we can lift a basis of $\mathrm{Ker}(d\otimes\kappa)$ along the surjection $\mathrm{Ker}(d)\to\mathrm{Ker}(d)\otimes\kappa\to\mathrm{Ker}(d\otimes\kappa)$ to elements $x_\bullet$ in $\mathrm{Ker}(d)$, and extend it by elements $y_\bullet$ in M whose residue classes are a basis of the quotient $(M\otimes\kappa)/\mathrm{Ker}(d\otimes\kappa)$. Since M is free of rank $\dim_\kappa(M\otimes\kappa)$, Nakayama's lemma implies that in this way we obtain a basis of M. The residue classes of the $y_\bullet$ then generate $M/\mathrm{Ker}(d)$. Since the images of the $y_\bullet$ are linearly independent in $N\otimes\kappa$, this proves the desired injectivity.

(ii) $\Rightarrow$ (i). Given (ii), after replacing R by R_f, the short exact sequence $0\to\mathrm{Im}(d)\to N\to\mathrm{Coker}(d)\to 0$ splits, so $\mathrm{Im}(d)$ is projective. Hence the short exact sequence $0\to\mathrm{Ker}(d)\to M\to\mathrm{Im}(d)\to 0$ splits, too, and thus the formation of $\mathrm{Ker}(d)$ commutes with arbitrary tensor product $-\otimes_R K$. In particular (i) holds (and the map there is even an isomorphism).

The final assertion is clear. $\square$

Jan. 17, 2024

Parts (3) and (4) of Theorem 4.3 follow from the following two propositions. For these propositions, the morphism f and the sheaf $\mathscr{F}$ do not directly play any role; we only need the complex $E^\bullet$ from Proposition 4.4. Given $E^\bullet$, we define functors T^i from the category of R-modules to itself by

$$T^i(M) := H^i(E^\bullet \otimes_R M).$$

Lemma 4.8. ([GW2], Lemma 23.120) *Let $E^\bullet$ be a complex of finitely generated projective R-modules.*

(1) *The collection $(T^i)_i$ is a δ-functor.*

(2) *For every R-module M, we have the* base change homomorphism

$$\beta^i(M) = T^i(R) \otimes_R M \to T^i(M).$$

(3) *The morphism β^i of functors is an isomorphism (i.e., $\beta^i(M)$ is an isomorphism for all M) if and only if T^i is right exact.*

Proof. Part (1) holds since all E^i are flat over R by assumption. Since T^i is an R-linear functor, we have a map

$$M = \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(T^i(R), T^i(M)),$$

so we obtain (2). For (3), note that T^i commutes with arbitrary direct sums, so that $\beta^i(M)$ is always an isomorphism for every free R-module M. If T^i is right exact, we get the same for arbitrary R-modules M by considering a presentation $F' \to F \to M \to 0$ with F, F' free. Conversely, use that the functor $T^i(R) \otimes -$ is right exact. $\square$

With $E^\bullet$ as in Proposition 4.4, the functor T^i can be written as $T^i(M) = H^i(X, \mathscr{F} \otimes_R M)$, and $\beta^i(M)$ is the base change homomorphism of Theorem 4.3.

Proposition 4.9. (cf. [GW2] Proposition 23.123)

Let R be a noetherian ring, and let $E^\bullet$ be a bounded complex of finite projective R-modules. Fix $i \in \mathbb{Z}$ and $s \in \operatorname{Spec} R$ such that the base change map

$$\beta^i(\kappa(s))\colon H^i(E^\bullet) \otimes \kappa(s) \to H^i(E^\bullet \otimes \kappa(s))$$

is surjective.

Then there exists $f \in R$ such that $s \in D(f)$ and such that after replacing R by R_f, $\beta^i(M)$ is an isomorphism for every R-module M, and the functor T^i is right exact.

Proof. After shrinking S, we may assume that all E^i are finite free R-modules. By the assumptions, $\mathrm{Ker}(d^i) \otimes \kappa(s) \to \mathrm{Ker}(d^i \otimes \kappa(s)) / \mathrm{Im}(d^{i-1} \otimes \kappa(s))$ is surjective. Since $\mathrm{Im}(d^{i-1}) \subseteq \mathrm{Ker}(d^i)$ and the maps $\mathrm{Im}(d^{i-1}) \to \mathrm{Im}(d^{i-1}) \otimes \kappa(s) \to \mathrm{Im}(d^{i-1} \otimes \kappa(s))$ are surjective, we even have that the map

$\mathrm{Ker}(d^i) \otimes \kappa(s) \to \mathrm{Ker}(d^i \otimes \kappa(s))$ is surjective. Thus by Lemma 4.7, after localizing R, if necessary, $\mathrm{Im}(d^i)$ and $\mathrm{Coker}(d^i)$ are free.

Since $\mathrm{Im}(d^i)$ is free, $\mathrm{Ker}(d^i)$ is a direct summand of E^i, so its formation commutes with $- \otimes M$. Thus the same is true for $H^i(E^\bullet \otimes M) = \mathrm{Coker}(E^{i-1} \to \mathrm{Ker}(d^i))$.

For the final statement, use Lemma 4.8. □

Proposition 4.10. ([GW2] Proposition 23.127)

Let $S = \operatorname{Spec} R$ be a noetherian affine scheme and let $E^\bullet$ be a bounded complex of coherent flat $\mathscr{O}_S$-modules. Let $s \in S$ such that $\beta^i(\kappa(s))$ is surjective.

Then $\beta^i(M)$ is an isomorphism for every R-module M. Moreover, the following are equivalent.
(i) *$\beta^{i-1}(\kappa(s))$ is surjective,*
(ii) *$H^i(E^\bullet)$ is locally free.*

Proof. By Proposition 4.9, after shrinking S, $\beta^i(M)$ is an isomorphism for every M and $T^i(-) = H^i(E^\bullet) \otimes -$ is right exact. It is left exact if and only if T^{i-1} is right exact since $(T^i)_i$ is a δ-functor (Lemma 4.8).

Now if $\beta^{i-1}(\kappa(s))$ is surjective, then (after shrinking S again, if necessary), also T^{i-1} is right exact, so T^i is exact. But this means that $H^i(E)$ is flat, and being finite over the noetherian ring R, it is locally free.

Conversely, if $H^i(E)$ is flat, then T^i is exact, so T^{i-1} is right exact and (using Lemma 4.8 once again) $\beta^{i-1}(M)$ is an isomorphism for all M, and in particular (i) holds. □

(4.3) Cohomology and base change, corollaries and applications.

Corollary 4.11. (Grauert's theorem) *Consider situation (S), assume that S is reduced and that the function $s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathscr{F}_s)$ is locally constant on S. Then $R^i f_* \mathscr{F}$ is a locally free $\mathscr{O}_S$-module (of finite rank), and the base change homomorphisms β^j are isomorphisms for $j = i, i-1$.*

Proof. We may assume that S is affine. Let $E^\bullet$ be as above. We may assume that all E^i are finite free R-modules. In view of (4.2.1), it follows that the maps

$$s \mapsto \dim \mathrm{Coker}(d^i \otimes \kappa(s)), \quad s \mapsto \dim \mathrm{Coker}(d^{i-1} \otimes \kappa(s)),$$

being lower semicontinuous with constant sum, are constant on S. Since S is reduced by assumption, this implies that $\mathrm{Coker}(d^i)$ and $\mathrm{Coker}(d^{i-1})$ are locally free $\mathscr{O}_S$-modules ([GW1] Corollary 11.19 or [Mu] §5 Lemma 1). This implies that the exact sequence

$$0 \to \mathrm{Im}(d^i) \otimes \mathscr{G} \to E^{i+1} \otimes \mathscr{G} \to \mathrm{Coker}(d^i) \otimes \mathscr{G} \to 0$$

splits for every quasi-coherent $\mathscr{O}_S$-module $\mathscr{G}$. In particular $\mathrm{Im}(d^i)$ is locally free, and $\mathrm{Im}(d^i \otimes \mathscr{G}) \cong \mathrm{Im}(d^i) \otimes \mathscr{G}$ (since the same holds for the cokernel).

From this, considering the short exact sequence

$$0 \to H^i(E) \otimes \mathscr{G} \to \mathrm{Coker}(d^{i-1}) \otimes \mathscr{G} \to \mathrm{Im}(d^i) \otimes \mathscr{G} \to 0$$

we obtain that $H^i(E)$ is locally free and that β^i is an isomorphism of functors. By Theorem 4.3, β^{i-1} is an isomorphism, as well. □

Corollary 4.12. *Let k be a field. Let $f\colon X \to S$ be a flat projective surjective morphism of k-schemes of finite type. Assume that all fibers of f are geometrically reduced and geometrically connected (i.e., denoting by $\overline{\kappa(s)}$ an algebraic closure of $\kappa(s)$, the scheme $X_s \otimes_{\kappa(s)} \overline{\kappa(s)}$ is reduced and connected for every $s \in S$). Then the natural homomorphism $\mathscr{O}_S \to f_*\mathscr{O}_X$ is an isomorphism.*

Proof. Problem sheet 12. □

(Conversely, it follows from the "Theorem on Formal Functions" that a proper morphism $f\colon X \to S$ such that $\mathscr{O}_S \cong f_*\mathscr{O}_X$ has geometrically connected fibers, see, e.g., [GW2] Corollary 24.50.)

5. Grassmannians, flag varieties, Schubert varieties

General references: [GW1] Ch. 8.

(5.1) The Grassmannian variety.

Recall that for a field k, $\mathbb{P}^n(k)$ is identified with the set of lines (i.e., one-dimensional linear subspaces) in k^{n+1}; in fact, we took that as a definition in *Algebraic Geometry 1.* For a description of $\mathbb{P}^n(S)$ for a general scheme S, one usually, following Grothendieck, prefers to talk about one-dimensional *quotients* rather than subspaces. Over a field one can switch back and forth between the two points of view by passing to dual vector spaces; this is even true over a general base scheme, since only locally free modules of finite rank are involved.

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The Grassmannian variety $\mathrm{Grass}_{r,n}$ attached natural numbers $0 \le r \le n$ similarly, but more generally, parameterizes r-dimensional linear subspaces of the n-dimensional vector space k^n. As for projective space, one can easily describe the functor represented by this scheme (and we will take this as our starting point); and also, the Grassmannian can be explicitly constructed by gluing copies of affine space in a suitable way.

Definition 5.1. *Let $0 \le r \le n$. We define the* Grassmann functor *by*

$$\mathrm{Grass}_{r,n}(S) = \{\mathscr{U} \subseteq \mathscr{O}_S^n;\ \mathscr{O}_S^n/\mathscr{U} \text{ is a locally free } \mathscr{O}_S\text{-module of rank } n-r\}.$$

We turn this into a functor as follows. For a morphism $S' \to S$ the map $\mathrm{Grass}_{r,n}(S) \to \mathrm{Grass}_{r,n}(S')$ is given by pullback, i.e., $\mathscr{U} \mapsto f^*\mathscr{U}$. Note that the natural map $f^*\mathscr{U} \to f^*\mathscr{O}_S^n = \mathscr{O}_{S'}^n$ is in fact injective (since $\mathscr{O}^n/\mathscr{U}$ is locally free, cf. Lemma 4.6) and that $\mathscr{O}_{S'}^n/f^*\mathscr{U} \cong f^*(\mathscr{O}_S^n/\mathscr{U})$ is locally free of rank $n-r$. The same lemma implies that $\mathscr{U}$ itself is locally free (of rank r).

For $r = n-1$ (and dually, for $r = 1$), this functor is represented by $\mathbb{P}^{n-1}$.

Our first goal is to show that for any choice of $r \le n$ this functor is representable by a scheme which admits an open cover by affine spaces. The open subsets of this cover admit a simple functorial description themselves (which of course is equivalent to the usual functorial description of affine space, but written a bit differently). This gives rise to “open subfunctors” of the Grassmann functor in the following sense.

Definition 5.2.

(1) *A morphism $F \to G$ of functors $(\mathrm{Sch})^{\mathrm{opp}} \to (\mathrm{Sets})$ is called* representable*, if for every scheme T (considered as a functor $(\mathrm{Sch})^{\mathrm{opp}} \to (\mathrm{Sets})$) and every morphism $T \to G$ the fiber product functor $F \times_G T$ is representable by a scheme.*

(2) *Given a morphism $f\colon F \to G$ of functors $(\mathrm{Sch})^{\mathrm{opp}} \to (\mathrm{Sets})$, we call F an* open subfunctor *of G, if the morphism f is representable by an open*

immersion, i.e., f is representable and for every scheme T and morphism $T \to G$ the scheme morphism $F \times_G T \to T$ is an open immersion.

(3) *Let $F\colon (\mathrm{Sch})^{\mathrm{opp}} \to (\mathrm{Sets})$ be a functor. A family $(F_i \to F)_i$ of open subfunctors of F is called an* open cover of the functor F*, if for every scheme T and morphism $T \to G$ the open immersions $F_i \times_F T \to T$ induce an open cover of T.*

Definition 5.3. *A functor $F\colon (\mathrm{Sch})^{\mathrm{opp}} \to (\mathrm{Sets})$ is a* sheaf for the Zariski topology*, if for every scheme S and open cover $S = \bigcup_i U_i$ the "sheaf sequence"*

$$F(S) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is exact (i.e., is an equalizer in the category of sets).

With this terminology, "gluing of morphisms" via the Yoneda lemma becomes the following proposition.

Proposition 5.4. *Every representable functor $F\colon (\mathrm{Sch})^{\mathrm{opp}} \to (\mathrm{Sets})$ is a sheaf for the Zariski topology.*

Proposition 5.5. ([GW1] Theorem 8.9) *Let $F\colon (\mathrm{Sch})^{\mathrm{opp}} \to (\mathrm{Sets})$ be a functor that has the following properties:*
(a) *F is a sheaf for the Zariski topology,*
(b) *F has an open cover by representable subfunctors.*
Then F is representable.

Sketch of proof. We construct a scheme representing the functor F by gluing the schemes representing the open subfunctors which cover F. The gluing datum (isomorphisms between the intersections satisfying the cocycle conditions) is obtained from the open cover of the functor F via the Yoneda lemma. E.g., if U_i represents F_i, then $F_i \times_F F_j$ is representable by an open subscheme U_{ij} of U_i which will correspond to the intersection of U_i and U_j in the result of the gluing process. Let X denote the scheme obtained by gluing.

Since F is a sheaf for the Zariski topology, the maps $U_i = F_i \to F$ can be glued to give a morphism $X \to F$. For every scheme S this gives rise to a bijection $X(S) \to F(S)$ (since this can be checked locally on S and hence reduces to a statement about every F_i), and thus is an isomorphism of functors, as desired. $\square$

Gluing of $\mathscr{O}_S$-modules implies that the Grassmann functor satisfies Property (a):

Lemma 5.6. *The Grassmann functor $\mathrm{Grass}_{r,n}$ is a sheaf for the Zariski topology.*

Definition 5.7. *Let $0 \le r \le n$. For $I \subseteq \{1, \dots, n\}$ of cardinality $\#I = n - r$ let $\mathscr{O}_S^I \subseteq \mathscr{O}_S^n$ denote the free $\mathscr{O}_S$-module generated by the standard basis* *vectors e_i, $i \in I$. We define*

$$\mathrm{Grass}^I_{r,n}(S) = \{\mathscr{U} \in \mathrm{Grass}_{r,n}(S);\ \textit{the composition } \mathscr{O}_S^I \hookrightarrow \mathscr{O}_S^n \to \mathscr{O}_S^n/\mathscr{U} \textit{ is an isomorphism}\}.$$

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By definition, $\mathrm{Grass}^I_{r,n}$ is a subfunctor of $\mathrm{Grass}_{r,n}$ and we claim that it is even an open subfunctor. Indeed, consider a scheme S and a morphism $S \to \mathrm{Grass}_{r,n}$ of functors. By the Yoneda lemma, it corresponds to a submodule $\mathscr{U} \subseteq \mathscr{O}_S^n$ with locally free quotient. Then the fiber product $\mathrm{Grass}^I_{r,n} \times_{\mathrm{Grass}_{r,n}} S$ is representable by the open subscheme

$$\{s \in S;\ \kappa(s)^I \to \kappa(s)^n \to \mathscr{O}_S^n/\mathscr{U}(s)\}$$

of S. Here, as usual, $\mathscr{U}(s) = \mathscr{U} \otimes_{\mathscr{O}_S} \kappa(s)$ denotes the fiber of $\mathscr{U}$ at s.

It also follows from this description that the open subfunctors $\mathrm{Grass}^I_{r,n}$ for all I give rise to an open cover of $\mathrm{Grass}_{r,n}$ since this can be checked on field-valued points.

Lemma 5.8. *For r, n, I as above, the functor $\mathrm{Grass}^I_{r,n}$ is representable by $\mathbb{A}^{r(n-r)}$.*

Proof. Write $J := \{1, \dots, n\} \setminus I = \{j_1 < \dots < j_r\}$. We view $\mathbb{A}^{r(n-r)}$ as the space of $(n \times r)$-matrices that have the unit matrix in rows $j_1, \dots, j_r$, and arbitrary entries in the other $n - r$ rows. For each such matrix (with entries in $\Gamma(S, \mathscr{O})$ for some scheme S), the subspace of $\mathscr{O}_S^n$ generated by the columns of the matrix is an element of $\mathrm{Grass}^I_{r,n}$, and this map defines an isomorphism of functors. In fact, it is clearly injective. To show surjectivity, take $\mathscr{U} \in \mathrm{Grass}^I(S)$ and consider the short exact sequence

$$0 \to \mathscr{U} \to \mathscr{O}_S^n \to \mathscr{O}_S^n/\mathscr{U} \to 0.$$

The composition $\mathscr{O}_S^n/\mathscr{U} \cong \mathscr{O}_S^I \subset \mathscr{O}_S^n$ defines a splitting of this sequence, and this induces an isomorphism $\mathscr{O}_S^J \cong \mathscr{O}_S^n/\mathscr{O}_S^I \cong \mathscr{U}$. In particular $\mathscr{U}$ is free and $\mathscr{U} \oplus \mathscr{O}_S^I = \mathscr{O}_S^n$, so $\mathscr{U}$ has a basis which can be extended to a basis of $\mathscr{O}_S^n$ by adding the standard basis vectors of $\mathscr{O}_S^I$. This means precisely that it lies in the image of the above map.

See [GW1] Lemma 8.13 for a different way of phrasing the proof. □

Putting everything together, we obtain:

Theorem 5.9. *The Grassmann functor $\mathrm{Grass}_{r,n}$ is representable by a scheme of finite type over $\mathbb{Z}$. It admits an open cover by copies of $\mathbb{A}^{r(n-r)}$, in particular it is smooth over $\mathbb{Z}$.*

(5.2) The Plücker embedding.

Let us show that the Grassmann scheme is projective. We will specify an explicit closed embedding $\mathrm{Grass}_{r,n} \to \mathbb{P}^N_{\mathbb{Z}}$, where $N = \binom{n}{r} - 1$, the so-called

Plücker embedding. Here we view $\mathbb{P}^N$ as the space of lines in $\bigwedge^r \mathscr{O}^n$, i.e., we use the "dual point of view" on the functorial description of projective space.

Theorem 5.10. ([GW1] Section (8.10)) *Let $0 \le r \le n$ and $N = \binom{n}{r} - 1$. The morphism $\mathrm{Grass}_{r,n} \to \mathbb{P}^N$ of functors given by*

$$\mathrm{Grass}_{r,n}(S) \to \mathbb{P}^N(S), \quad \mathscr{U} \mapsto \left(\bigwedge^r \mathscr{U} \subseteq \bigwedge^r \mathscr{O}_S^n\right)$$

gives rise to a closed immersion of schemes.

Remarks on the proof. We mostly skip the proof here. In terms of coordinates, the map maps a (free) r-dimensional subspace, with basis the columns of some $(n \times r)$-matrix A, say, to the vector with homogeneous coordinates all the r-minors (i.e., determinants of $(r \times r)$-submatrices) of A. It is easy to show that the morphism $\mathrm{Grass}_{r,n} \to \mathbb{P}^N$ is injective (on the underlying sets). Given an element of $\mathbb{P}^N(S)$ which we view as a locally free line bundle in $\bigwedge^r \mathscr{O}_S^n$ with locally free quotient, we construct a map $\varphi_{\mathscr{L}} \colon \mathscr{O}_S^n \to \bigwedge^{r+1} \mathscr{O}_S^n$ as follows. Locally on S, $\mathscr{L}$ is free, say with basis the vector $v \in \bigwedge^r \mathscr{O}_S^n$. Then we map $w \mapsto w \wedge v$. The condition that the kernel of $\varphi_{\mathscr{L}}$ is a locally free submodule of $\mathscr{O}^n$ of rank r with locally free quotient is a closed condition on $\mathbb{P}^N$. The image of the morphism from the Grassmannian is contained in the closed subscheme of $\mathbb{P}^N$ defined in this way, and there the above morphism admits the inverse $\mathscr{L} \mapsto \mathrm{Ker}(\varphi_{\mathscr{L}})$. □

We obtain the following corollary, which can also easily be shown directly via the valuative criterion for properness (Theorem 1.2).

Corollary 5.11. *Let $0 \le r \le n$ and $N = \binom{n}{r} - 1$. The Grassmann scheme $\mathrm{Grass}_{r,n}$ is proper over $\mathrm{Spec}(\mathbb{Z})$.*

(5.3) Canonical bundle and Picard group of the Grassmannian.

Fix $0 \le r \le n$. The definition of the Grassmannian $\mathrm{Gr} = \mathrm{Grass}_{r,n}$ as representing a functor provides us with a universal object, i.e., an $\mathscr{O}_{\mathrm{Gr}}$-submodule $\mathscr{U} \subseteq \mathscr{O}_{\mathrm{Gr}}^n$ of rank r with locally free quotient.

We obtain a short exact sequence

$$0 \to \mathscr{U} \to \mathscr{O}_{\mathrm{Gr}}^n \to \mathscr{Q} \to 0$$

of locally free $\mathscr{O}_{\mathrm{Gr}}$-modules. It is then straightforward to translate the proof of Proposition 2.46 (about the Euler sequence on projective space) to prove the following generalization.

Proposition 5.12. ([GW2] Theorem 17.46) *We have an isomorphism*

$$\Omega_{\mathrm{Gr}/\mathbb{Z}} \cong \mathscr{H}om_{\mathscr{O}_{\mathrm{Gr}}}(\mathscr{Q}, \mathscr{U}).$$

From this we can easily compute the canonical bundle, i.e., the highest exterior power of the sheaf of differentials. This is also the dualizing sheaf of the Grassmannian (after base change to a field in order to get into the setting where we defined this).

Corollary 5.13. *With notation as above, let* $\omega_{\mathrm{Gr}} := \bigwedge^{r(n-r)} \Omega_{\mathrm{Gr}/\mathbb{Z}}$ *be the canonical bundle of the Grassmannian. Then*

$$\omega_{\mathrm{Gr}} \cong \left(\bigwedge^r \mathscr{U}\right)^{\otimes n}$$

Proof. The computation of $\Omega_{\mathrm{Gr}/\mathbb{Z}}$ give us, denoting the highest exterior power of a locally free sheaf $\mathscr{E}$ by $\det(\mathscr{E})$,

$$\begin{aligned}\omega_{\mathrm{Gr}} &\cong \det\left(\mathscr{Q}^\vee \otimes \mathscr{U}\right)\\ &\cong \det(\mathscr{Q})^{\vee,\otimes r} \otimes \det(\mathscr{U})^{\otimes(n-r)}\\ &\cong \det\left(\mathscr{U}\right)^{\otimes n},\end{aligned}$$

where in the second step we use the lemma below, and for the final step we have used that (as a result of the short exact sequence defining $\mathscr{Q}$) $\det(\mathscr{Q})^\vee \cong \det(\mathscr{U})$. □

Lemma 5.14. *Le* X *be a ringed space and let* $\mathscr{E}$, $\mathscr{F}$ *be locally free sheaves on* X *of finite ranks* e *and* f, *respectively. Write* $\det(-)$ *for the highest exterior power of a locally free sheaf of finite rank. Then*

$$\det(\mathscr{E} \otimes \mathscr{F}) \cong \det(\mathscr{E})^{\otimes f} \otimes_{\mathscr{O}_X} \det(\mathscr{F})^{\otimes e}.$$

Sketch of proof. Locally $\mathscr{E}$ and $\mathscr{F}$ are free, with bases $(b_i)_i$ and $(c_j)_j$, say. Then $(b_i \otimes c_j)_{i,j}$ is a basis of $\mathscr{E} \otimes \mathscr{F}$, and we obtain a homomorphism $\det(\mathscr{E} \otimes \mathscr{F}) \to \det(\mathscr{E})^{\otimes f} \otimes_{\mathscr{O}_X} \det(\mathscr{F})^{\otimes e}$ by mapping

$$(b_1 \otimes c_1) \wedge (b_1 \otimes c_2) \wedge \cdots (b_e \otimes c_f) \mapsto (b_1 \wedge \cdots \wedge b_e)^{\otimes f} \otimes (c_1 \wedge \cdots c_f)^{\otimes e}.$$

Since it maps a basis vector to a basis vector (source and target are free of rank 1), it is an isomorphism. One checks that this homomorphism is independent of the choices of bases. Thus the local homomorphisms glue. □

One can show that the Picard group of the Grassmannian (over a field, or more generally over any unique factorization domain) is isomorphic to $\mathbb{Z}$, more precisely we have the following result.

Proposition 5.15. *Let* k *be a field, and let* $0 < r < n$. *Let* $\mathrm{Grass}_{r,n}$ *be the Grassmannian of* r*-dimensional subspaces in* n*-dimensional space over* k, *and let* $\mathscr{U}$ *denote the universal object as above. Then pullback under the Plücker embedding maps* $\mathscr{O}(-1)$ *to* $\bigwedge^r \mathscr{U}$ *and induces an isomorphism* $\mathrm{Pic}(\mathrm{Grass}_{r,n}) \cong \mathrm{Pic}(\mathbb{P}^N_k) = \mathbb{Z}$.

Remarks on the proof. Consider one of the open charts $\mathrm{Grass}^I_{r,n}$ of Gr. The key point is to show that its complement is irreducible of codimension 1, i.e., can be considered as a primitive Weil divisor. This follows from the analysis of “Schubert varieties” in the Grassmannian (which we omit here, but see below for “Schubert varieties in flag varieties”).

Then [GW1] Proposition 11.42 and the fact that $\mathrm{Pic}(\mathrm{Grass}^I_{r,n}) = 0$ (since $\mathrm{Grass}^I_{r,n}$ is isomorphic to affine space) provide a surjection $\mathbb{Z} \to \mathrm{Pic}(\mathrm{Grass}_{r,n})$. This maps 1 to the pullback of $\mathscr{O}(1)$ under the Plücker embedding. Since $\mathrm{Grass}_{r,n}$ is projective of positive dimension, the Picard group cannot be finite; cf. [GW1] Corollary 13.82 for one way to show this. □

(5.4) The flag variety.

Recall that a *(full) flag* in a finite-dimensional vector space V over some field k is a chain $0 \subset F_1 \subset \cdots \subset F_{\dim V - 1} \subset V$ of subspaces with $\dim F_i = i$ for all i. Jan. 29, 2024

Proposition/Definition 5.16. *Let $n \geq 1$.*

(1) *The full flag functor* $(\mathrm{Sch})^{\mathrm{opp}} \to (\mathrm{Sets})$,

$$\mathrm{Flag}(S) = \left\{ (F_i)_i \in \prod_{i=1}^{n-1} \mathrm{Grass}_{i,n}(S);\ \forall i\colon F_i \subset F_{i+1} \right\}$$

is representable by a closed subscheme of $\prod_{i=1}^{n-1} \mathrm{Grass}_{i,n}(S)$ *which we also denote by* Flag. *The scheme* Flag *is projective over* $\mathbb{Z}$.

(2) *Similarly, given $0 < i_1 < \cdots < i_r < n$, we have the* partial flag variety $\mathrm{Flag}_{i_\bullet}$ *of all partial flags $(F_\nu)_\nu$ with $F_\nu \in \mathrm{Grass}_{i_\nu,n}$.*

Sketch of proof. To show that the full flag functor is representable by a closed subscheme of the product of Grassmannians we need to show that the conditions $F_i \subset F_{i+1}$ are “closed conditions”. By induction we reduce to the following

Claim. The subfunctor

$$S \mapsto \{(F, F') \in \mathrm{Grass}_{r,n}(S) \times \mathrm{Grass}_{r',n}(S);\ F \subseteq F'\}$$

is represented by a closed subscheme of the product $\mathrm{Grass}_{r,n} \times \mathrm{Grass}_{r',n}$.

Proof of claim. We can check this locally on the product and therefore pass to a product $\mathrm{Grass}^I_{r,n} \times \mathrm{Grass}^{I'}_{r',n}$ for suitable I, I' (with notation as in Definition 5.7). Now given $(F, F') \in \mathrm{Grass}^I_{r,n} \times \mathrm{Grass}^{I'}_{r',n}$, F and F' are given by matrices M, M' as in the proof of Lemma 5.8. The condition $F \subseteq F'$ is then equivalent to the existence of a matrix $A \in \mathrm{Mat}_{r' \times r}(\Gamma(S, \mathscr{O}_S))$ such that $M = M'A$. But since M' has the unit matrix in rows $J' := \{1, \dots, n\} \setminus J'$, there is at most one choice for A, namely the submatrix $M_{J',\bullet}$ of M in rows J'. The condition $F \subseteq F'$ thus translates to $M_{I',\bullet} = M'_{I',\bullet} M_{J',\bullet}$. This is clearly a closed condition. □

Proposition 5.17. *Fix a non-empty set $I = \{i_1 < \cdots < i_s\} \subseteq \{1, \dots, n\}$. The (partial) flag variety* Flag_I *admits an open cover by affine spaces. It is* *a smooth projective $\mathbb{Z}$-scheme of relative dimension $i_1(n-i_1)+(i_2-i_1)(n-i_2)+\cdots+(i_s-i_{s-1})(n-i_s)$.*

Proof. We have already proved the projectivity, so it remains to exhibit an open cover by affine spaces and compute the dimension. To simplify the notation, we give the proof for the full flag variety only.

Consider the free module $\mathbb{Z}^n$ and denote by $e_1,\dots,e_{n-1}$ its standard basis. For a permutation $w\in S_n$ let $G^w_\bullet$ be the flag defined as

$$G^w_i = \langle e_{w(n)},\dots,e_{w(n-i+1)}\rangle.$$

For any scheme S by pullback we obtain a flag in $\mathscr{O}_S^n$ which we again denote by G^w_i. We define

$$\mathrm{Flag}^w(S) := \{(F_i)_i\in \mathrm{Flag}(S);\ \forall i\colon G^w_{n-i}\to \mathscr{O}_S^n/F_i \text{ is an isomorphism}\}.$$

This is the intersection (in $\prod \mathrm{Grass}_{i,n}$) of Flag with a product of standard (up to permutation of the basis) open charts of Grassmannians and hence an open subfunctor of Flag, represented by an open subscheme of Flag.

It is now enough to show that each Flag^w is isomorphic to an affine space of dimension $n(n-1)/2$ and that the Flag^w cover Flag. This latter property can be checked on field-valued points where it becomes a simple linear algebra consideration.

To show that Flag^w is an affine space (of the correct dimension), by renumbering our basis we may assume that $w=\mathrm{id}$. Denote by U^- the scheme of lower triangular matrices whose diagonal entries are all $=1$. The scheme U^- is obviously isomorphic to $\mathbb{A}^{n(n-1)/2}$. One then shows, similarly as in the proof of Lemma 5.8 that the map $U^-\to\mathrm{Flag}^{\mathrm{id}}$ which maps a matrix $A\in U^-(S)$ to the flag $(F_i)_i$ where F_i is the free $\mathscr{O}_S$-submodule of $\mathscr{O}_S^n$ generated by the first i columns of A, is an isomorphism.

(For general w, one obtains an isomorphism $U^-\to\mathrm{Flag}^w$ by mapping $A\in U^-(S)$ to the flag $(F_i)_i$ where F_i is generated by the first i columns of wA, where we view w as a permutation matrix.) □

Remark 5.18. Let k be a field and denote by Flag the flag variety over k, for some fixed n. The group $GL_n(k)$ of invertible $(n\times n)$-matrices with entries in k acts transitively on $\mathrm{Flag}(k)$, and denoting by $E_\bullet$ the standard flag in k^n, the map $g\mapsto gE_\bullet$ induces a bijection $GL_n(k)/B(k)\xrightarrow{\sim}\mathrm{Flag}(k)$, where $B(k)\subseteq GL_n(k)$ denotes the stabilizer of the standard flag (i.e., the subgroup of upper triangular matrices).

The action is given by a morphism $GL_{n,k}\times_{\mathrm{Spec}\,k}\mathrm{Flag}\to\mathrm{Flag}$. One can make sense of the quotient GL_n/B in algebraic geometry, but (as usual) dealing with quotients is more complicated than in differential or complex geometry, for instance, because the Zariski topology is too coarse.

(5.5) The Gauß-Bruhat algorithm.

Fix a field k and a natural number $n \geq 1$. We consider full flags $F_\bullet$ in k^n. The *standard flag* E is the flag with i-th entry the subspace $\langle e_1, \dots, e_i \rangle$ generated by the first i standard basis vectors. Let S_n denote the symmetric groups on n letters. In this context, it is natural to view S_n as the subgroup of permutation matrices of $GL_n(k)$.

Recall that S_n is generated by the *simple transpositions*, i.e., by the transpositions $s_i = (i, i+1)$ of two adjacent elements ($i = 1, \dots, n-1$). We denote by $\mathbb{S}$ the set of simple transpositions.

Proposition 5.19. *Let $w \in S_n$ be a permutation. Then the number*

$$\ell(w) := \#\{(i,j);\ i < j,\ w(i) > w(j)\}$$

of inversions *of w is equal to the minimal number of factors in a product of simple transpositions that equals w. We call this number the* length *of w.*

Proof. One shows that for $w \in S_n$ and $s \in \mathbb{S}$, $\ell(ws) \geq \ell(w) - 1$ (more precisely, one can show that $\ell(ws) \in \{\ell(w)-1, \ell(w)+1\}$. And moreover that for every $w \in S_n$ with $\ell(w) > 0$ there exists $s \in \mathbb{S}$ with $\ell(ws) < \ell(w)$. Both statements follow from elementary considerations about permutations. $\square$

Lemma 5.20. (Gauß-Bruhat algorithm) *Let k be a field, $n \geq 1$, $G = GL_n(k)$ the group of invertible matrices of size $n \times n$ over k. Let $B \subset G$ be the subgroup of upper triangular matrices, and $U \subset B$ the subgroup of unipotent upper triangular matrices (i.e., upper triangular matrices all of whose diagonal entries are equal to 1). Let U^- be the group of unipotent lower triangular matrices in G. (In this lemma, we consider all these groups as abstract groups, rather than k-schemes/k-varieties.)*

(1) *We have $G = \bigcup_w BwB$ (disjoint union of B-double cosets). This is called the* Bruhat decomposition *of $GL_n(k)$.*
(2) *Let $w \in S_n$. Denote by U_w the following subgroup of the group of unipotent upper triangular matrices U:*

$$U_w = U \cap wU^-w^{-1}$$

(where we view w inside GL_n as a permutation matrix). The map $U_w \times B \xrightarrow{\sim} BwB$, $(u, b) \mapsto uwb$, is bijective.
(3) *The decomposition of $GL_n/B =$ Flag(k) as a disjoint union of B-orbits (for the B-action on GL_n/B by multiplication on the left, equivalently the action on* Flag(k) *induced by the standard action of B on k^n) is $GL_n/B = \bigcup_w BwE$ (where E, as usual, denotes the standard flag, which under the identification with GL_n/B is the residue class of the unit matrix). The bijections of Part (2) induce bijections $BwE \leftrightarrow U_w$, and U_w is in bijection (as a set) with $k^{\ell(w)}$. This decomposition is called the* Bruhat decomposition *of the space of flags* Flag(k).

Proof. The key point is that every matrix $g \in G$ can be written as $g = uwb$ for $u \in U$, $b \in B$, w a permutations matrix. Equivalently, given g, we can transform g to a permutation matrix by the following elementary row and column operations:

(1) Add a multiple of a row of the matrix g to another row *lying above* the given one.
(2) Add a multiple of a column of the matrix g to another column *lying to the right of* the given one.
(3) Multiply a column with a non-zero scalar.

This can be checked fairly easily by "directly manipulating matrices". $\square$

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Looking carefully at the proof of the previous lemma, we obtain the following "scheme-theoretic version" of the Gauß-Bruhat algorithm.

Proposition 5.21. (Gauß-Bruhat algorithm, scheme version) *Let k be a field, $n \geq 1$, $G = GL_{n,k}$ the k-group scheme of invertible matrices of size $n \times n$. Let $B \subset G$ be the subgroup scheme of upper triangular matrices, and $U \subset B$ the subgroup scheme of unipotent upper triangular matrices (i.e., upper triangular matrices all of whose diagonal entries are equal to 1). Similarly, let U^- be the scheme of unipotent lower triangular matrices.*

(1) *Let $w \in W$. The image of the morphism $B \times B \to G$ that is given on R-valued points (R a k-algebra) by $(b, b') \mapsto bwb'$ is locally closed. We denote by BwB the reduced subscheme of G whose topological space is this locally closed subset.*
(2) *We have $GL_n = \bigcup_w BwB$ (disjoint union of subschemes), i.e., every point of the topological space GL_n lies in exactly one of the subschemes BwB. This decomposition is called the* Bruhat decomposition *of the scheme GL_n.*
(3) *Let $w \in S_n$. Denote by U_w the following subgroup scheme of U:*

$$U_w = U \cap wU^- w^{-1}.$$

The morphism $U_w \times B \xrightarrow{\sim} BwB$ given on R-valued points by $(u, b) \mapsto uwb$, is an isomorphism of k-schemes.
(4) *The decomposition in (1) induces a decomposition of the flag variety* Flag *over k as a disjoint union of locally closed subschemes $C(w)$, where $C(w)$ is the reduced subscheme of* Flag *whose topological space is the image of BwB under the natural morphism $G \to \text{Flag}$.*

The isomorphisms in Part (2) induce isomorphisms $C(w) \cong U_w \cong \mathbb{A}_k^{\ell(w)}$.

The subscheme $C(w)$ of Flag *is called the* Schubert cell *attached to w. This decomposition is called the* Bruhat decomposition *of* Flag.

We can make the partition of the flag variety into Schubert cells very explicit using the notion of *relative position* of flags in the following sense.

Definition 5.22. *Let k be a field. For flags $F_\bullet$, $F'_\bullet$ in k^n we write*

$$\mathrm{inv}(F_\bullet, F'_\bullet) = w \in S_n,$$

if for all i, j we have

$$\dim(F_i \cap F'_j) = \#\{\nu \in \{1, \dots, j\};\ w(\nu) \le i\}$$

and call w the relative position *of the flags $F_\bullet$ and $F'_\bullet$.*

It follows from the Bruhat decomposition of $GL_n(k)$ that for any two flags, there exists a unique element of S_n with the properties of the above definition. We obtain a map

$$\mathrm{inv}\colon \mathrm{Flag}(k) \times \mathrm{Flag}(k) \to S_n.$$

Example 5.23. Let $F_\bullet$, $F'_\bullet$ be flags. Then $\mathrm{inv}(F_\bullet, F'_\bullet) = \mathrm{id}$ if and only if $F_\bullet = F'_\bullet$.

Let s be a simple transposition, i.e., $s = (i, i+1)$ for some $i \in \{1, \dots, n-1\}$. Then $\mathrm{inf}(F_\bullet, F'_\bullet) = s$ if and only if $F_j = F'_j$ for all $j \neq i$, and $F_i \neq F'_i$.

It is clear that the relative position with the standard flag $E_\bullet$ is constant on each Schubert cell. Thus basically by definition,

$$C(w)(k) = \{F \in \mathrm{Flag}(k);\ \mathrm{inv}(E_\bullet, F) = w\}.$$

Remark 5.24.
(1) Let k be a field, and let Flag denote the flag variety over k for some fixed n. The decomposition $\mathrm{Flag}(k) = \bigcup C(w)(k)$ is precisely the decomposition into $B(k)$-orbits for the natural action by $B(k)$ (the group of upper triangular matrices) on $\mathrm{Flag}(k)$, namely the action induced by the standard action of $B(k) \subseteq GL_n(k)$ on k^n.
(2) The theory of Schubert cells (and Schubert varieties as defined below, ...) can be generalized to other "reductive groups" (including, for instance, (special) orthogonal groups, symplectic groups, unitary groups). See e.g. [B], [Mi].

(5.6) Schubert varieties.

Let k be a field. We denote by Flag, $C(w)$ the flag variety and Schubert cells over k.

Definition 5.25. *Let $w \in S_n$. The* Schubert variety $X(w)$ *attached to w is the reduced closed subscheme of* Flag *whose underlying space is the closure of $C(w)$.*

From the definition (and the projectivity of the flag variety) it is clear that $X(w)$ is a projective k-scheme. In general, $X(w)$ is not smooth and in fact Schubert varieties are a class of interesting varieties which are fairly explicit but have a "non-trivial" geometry.

Example 5.26. We have $X(\mathrm{id}) = C(\mathrm{id}) \cong \mathrm{Spec}(k)$. For $s \in \mathbb{S}$, $X(s) \cong \mathbb{P}^1_k$. For the longest element $w_0 \in W$, $C(w_0)$ is (open and) dense in Flag, whence $X(w_0) = \mathrm{Flag}$.

One can show that each $X(w)$ is a union of Schubert cells (cf. Remark 5.24). This defines a unique partial order on S_n which satisfies

$$X(w) = \bigcup_{v \le w} C(v),$$

the so-called *Bruhat order*. It follows immediately from the definition that $v \le w$ implies $\ell(v) \le \ell(w)$. The converse implication is not true. The Bruhat order can also be described in combinatorial terms, but the description is non-trivial because of the inherent complexity of the situation. The identity element is the unique minimal element and the unique element of maximal length, $w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$, is the unique maximal element with respect to the Bruhat order. The Schubert cell $C(w_0)$ is the unique open Schubert cell. It is equal to the open subscheme Flag^{w_0} defined in the proof of Proposition 5.17. Except for $n \le 2$ the Bruhat order is not a total order.

(5.7) The Demazure resolution.

We work over a fixed field k (i.e., Flag denotes the flag variety over k, ...).

Lemma 5.27. *Let $w \in S_n$ and $s \in \mathbb{S}$ such that $\ell(ws) > \ell(w)$. Let $F_i \in \mathrm{Flag}(k)$.*
(1) *If $\mathrm{inv}(F_1, F_2) = w$, $\mathrm{inv}(F_2, F_3) = s$, then $\mathrm{inv}(F_1, F_3) = ws$.*
(2) *Given F_1, F_3 with $\mathrm{inv}(F_1, F_3) = ws$, there exists a unique F_2 with $\mathrm{inv}(F_1, F_2) = w$, $\mathrm{inv}(F_2, F_3) = s$.*

Proof. The proof is "just linear algebra" and more or less straightforward. Recall that $\mathrm{inv}(F, F') = s_i$ is equivalent to saying that $F_j = F'_j$ for all $j \ne i$ and $F_i \ne F'_i$. □

From the lemma we obtain the following scheme-theoretic statement.

Proposition 5.28. *Let k be a field. Let $w \in S_n$ and $s \in \mathbb{S}$ such that $\ell(ws) > \ell(w)$.*

There is a (unique) morphism $C(ws) \to C(w)$ which on k'-valued points (for any extension field k' of k) is given by mapping a flag $F \in C(ws)(k)$ to the unique flag F' with $\mathrm{inv}(E, F') = w$, $\mathrm{inv}(F', F) = s$. (As usual, E denotes the standard flag.) Each fiber is isomorphic to the affine line $\mathbb{A}^1$.

Let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression, i.e., $s_\nu \in \mathbb{S}$, $\ell = \ell(w)$ (the minimal number of factors needed to express w as a product of elements of $\mathbb{S}$). Let $D(i_\bullet) := D(i_1, \dots, i_\ell)$ be the unique reduced closed subscheme of $\prod_\nu \mathrm{Flag}^\ell$ such that for every k'/k we have

$$D(i_1, \dots, i_\ell)(k') = \left\{ (F_\nu)_{\nu=1,\dots,\ell} \in \prod_\nu \mathrm{Flag}^\ell(k');\ \forall \nu\colon\ \mathrm{inv}(F_{\nu-1}, F_\nu) \in \{\mathrm{id}, s_{i_\nu}\} \right\}.$$

To see that this definition defines is a closed subset, recall that the condition $\mathrm{inv}(F, F') \in \{\mathrm{id}, s_i\}$ is equivalent to asking that $F_j = F'_j$ for all $j \neq i$.

Here we denote by $F_0 := E$ the standard flag.

Similarly, we define

$$D(i_1, \dots, i_\ell)^\circ = \left\{ (F_\nu)_{\nu=1,\dots,\ell} \in \prod_\nu \mathrm{Flag}^\ell;\ \forall \nu\colon\ \mathrm{inv}(F_{\nu-1}, F_\nu) = s_{i_\nu} \right\}.$$

This defines an open subscheme of $D(i_\bullet)$.

We denote by $\pi := \pi_{i_\bullet}\colon D(i_\bullet) \to \mathrm{Flag}$ the projection to the last factor, i.e., the map $(F_\nu)_\nu \mapsto F_\ell$.

Proposition 5.29. *With notation as above*

(1) *The natural projections*

$$D(i_1, \dots, i_\ell) \to D(i_1, \dots, i_{\ell-1}) \to \cdots \to D(i_1) \to \mathrm{Spec}(k)$$

are Zariski-locally trivial $\mathbb{P}^1_k$-bundles (i.e., Zariski-locally on the target of each morphism the source is isomorphic to a product with $\mathbb{P}^1_k$).

(2) *The scheme $D(i_1, \dots, i_\ell)$ is a smooth projective k-scheme*

(3) *The restriction to $\pi^{-1}(C(w))$ induces an isomorphism $\mathbb{A}^{\ell(w)} \cong D(i_\bullet)^\circ \xrightarrow{\sim} C(w)$.*

(4) *The map $\pi_{i_\bullet}$ has image $X(w)$.*

Proof. Part (1) is clear on k-valued points and with a bit more effort can be checked to be valid scheme-theoretically. Part (2) follows immediately from (1). Part (3) follows from Lemma 5.27.

Since Flag is proper, so is the closed subscheme $D(i_\bullet) \subset \prod \mathrm{Flag}$. Hence the morphism $D(i_\bullet) \to \mathrm{Flag}$ is proper and in particular has closed image. Since $D(i_\ell)$ and $X(w)$ are reduced, Part (4) follows from (3). □

In particular, $\pi\colon D(i_\bullet) \to X(w)$ is a "resolution of singularities" of $X(w)$, i.e., a surjective proper birational map onto $X(w)$ with smooth source. It is called the *Demazure resolution* (or the *Bott-Samelson resolution*).

(5.8) Demazure resolutions are rational resolutions.

Using the theory of "Frobenius splittings" (see [Ra], [BK]) one can show the following result.

Proposition 5.30. *The morphism $\pi := \pi_{i_\bullet}\colon D(i_\ell) \to X(w)$ is a* rational resolution *of $X(w)$, i.e.,*

(a) $\pi_* \mathscr{O}_{D(i_\bullet)} = \mathscr{O}_{X(w)}$,

(b) $R^q \pi_* \mathscr{O}_{D(i_\bullet)} = 0$ *for all* $q > 0$,

(c) $R^q \pi_* \omega_{D(i_\bullet)} = 0$ *for all* $q > 0$.

Corollary 5.31. *The Schubert variety $X(w)$ is Cohen-Macaulay.*

Proof. Problem 43. (See [Ra].) □

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