# ALGEBRAIC GEOMETRY 1, WINTER TERM 2025/26. LECTURE NOTES.

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## 1. Introduction

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We will start with a relatively long *introductory chapter*, in order to ...

- provide some motivation for the (partly more technical) content that will come later,
- give those participants who were not in the Algebra 2 class last term a little more time to brush up their commutative algebra knowledge:
  - (Prime) ideals, quotients
  - localization (with respect to a multiplicative subset; in particular with respect to one element and localization at a prime ideal),
  - spectrum of a ring, Zariski topology (this we will redo in the class, but ideally you are already a little familiar with the notion of *topological space*.

I will try to address the question *What is algebraic geometry?*, and at the same time give, towards the end of the chapter, a rough survey of this class.

### (1.1) What Algebraic Geometry is about.

In one sentence: Study "geometric properties" of solution sets of systems of polynomial equations (over a field, or more generally a commutative ring).

**Example 1.1.**

$$\{(x, y) \in \mathbb{R}^2 \mid y^2 = x^2(x + 1)\} \quad \subset \mathbb{R}^2.$$

**Comparison with Previous/Other Courses**

| | |
|---|---|
| Linear Algebra | Systems of *linear* equations |
| Algebra | polynomial equations (1 variable, 1 polynomial) |
| Algebraic Geometry | systems of polynomial equations |
| Algebraic Number Theory | ...coefficients/solutions in $\mathbb{Z}, \mathbb{Q}, K/\mathbb{Q}$ fin., $\mathbb{F}_q$ |

Here *algebraic* refers to the fact that we

- study solution sets (zero sets) of polynomials (not power series, differential/holomorphic functions, etc.),

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- use algebraic methods (specifically commutative algebra) to study these objects.

In particular, at least in principle, we may hence work over an arbitrary field (not only $\mathbb{R}$ or $\mathbb{C}$).

### (1.2) The Cayley–Hamilton Theorem: A Geometric View.

**Theorem 1.2.** [Cayley–Hamilton] *Let $k$ be a field, $A \in M_n(k)$. Then* $\operatorname{charpol}_A(A) = 0$.

We want to look at this result from the perspective of an algebraic geometer, i.e., we view $M_n(k)$ as $n$-dimensional (vector) space.

Let us consider the case $k = \mathbb{R}$, $n = 2$ and restrict to matrices $A$ with trace $\operatorname{tr}(A) = 0$. (This does not change the main argument, but simplifies the discussion a little bit and will allow us to draw a picture later.)

We want to use that the theorem is obviously true, if $A$ is a diagonal matrix. From this, it follows easily that the theorem holds whenever $A$ is diagonalizable. In fact, if $A = SDS^{-1}$ for a diagonal matrix $D$, then $\operatorname{charpol}_A = \operatorname{charpol}_D$. Since conjugation is a ring automorphism of the ring of matrices (over any ring), we may "pull it out" of any polynomial. Together we obtain

$$\operatorname{charpol}_A(A) = \operatorname{charpol}_D(SAS^{-1}) = S \operatorname{charpol}_D(D) S^{-1},$$

and the term on the right vanishes, since $\operatorname{charpol}_D(D) = 0$ by the case of diagonal matrices. Furthermore, in this argument we may just as well allow matrices $S$ with entries in some extension field of $k$, and we see that it suffices to assume that $A$ is *diagonalizable over* $\mathbb{C}$. But of course, there are also non-diagonalizable matrices.

So we consider a matrix

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M_2(\mathbb{R})^{\operatorname{tr}=0} \cong \mathbb{R}^3,$$

where we use $a$, $b$, $c$ as coordinates on $\mathbb{R}^3$. We then have

$$\operatorname{charpol}_A = (T - a)(T + a) - bc = T^2 - (a^2 + bc).$$

In particular we see that all matrices $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2 + bc \neq 0$ are diagonalizable over $\mathbb{C}$. On the other hand, if $a^2 + bc = 0$, then $A$ is not necessarily diagonalizable.

We now consider the map:

$$\chi : M_2(\mathbb{R})^{\operatorname{tr}=0} \to M_2(\mathbb{R}), \quad A \mapsto \operatorname{charpol}_A.$$

Our goal is to show that the map $\chi$ is constant with image the zero matrix. By what we have said, $\chi(A) = 0$ for all those $A$ that are diagonalizable over $\mathbb{C}$.

Since the map $\chi\colon \mathbb{R}^3 \to \mathbb{R}^4$ is given by polynomials, it is continuous. Therefore for every closed subset of $\mathbb{R}^4$, its inverse image under $\chi$ is again closed. We apply this to the set $\{0\}$ containing only the zero matrix; clearly this is a closed set. Its inverse image contains, by what we know already, all those traceless matrices that are diagonalizable over $\mathbb{C}$, and in particular all matrices $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2 + bc \neq 0$. But this set is dense in $M_2(\mathbb{R})^{\mathrm{tr}=0}$, i.e., its closure is the whole space. It follows that $\chi^{-1}(\{0\}) = M_2(\mathbb{R})^{\mathrm{tr}=0}$, as we wanted to show.

The same argument, with small modifications, applies when we drop the condition on the trace, and also for square matrices of arbitrary size.

**Question:** How to deal with other fields?
For this, we need a *notion of continuous map* in a more general context.

### (1.3) The Zariski topology on $k^n$.

Let $k$ be a field. Since we want to study solution sets of systems of polynomial equations, we introduce the following notation:

**Definition 1.3.**

(1) *Given* $f_1, \dots, f_m \in k[T_1, \dots, T_n]$, *we define the* vanishing set *(in German:* Verschwindungsmenge*)*
$$V(f_1, \dots, f_m) := \{(t_i) \in k^n;\ f_j(t_1, \dots, t_n) = 0\ \forall j\}.$$

(2) *More generally, for any subset* $\mathcal{F} \subset k[T_1, \dots, T_n]$, *we define the* vanishing set *of* $\mathcal{F}$ *as*
$$V(\mathcal{F}) = \{(t_i) \in k^n;\ f(t_1, \dots, t_n) = 0\ \forall f \in \mathcal{F}\}.$$

If $k'/k$ is a field extension, then we set
$$V(f_1, \dots, f_m)(k') := \{(t_i) \in (k')^n;\ f_j(t_i) = 0\ \forall j\},$$
and analogously define $V(\mathcal{F})(k')$.

**Remark 1.4.** Let $\mathcal{F} \subset k[T_1, \dots, T_n]$, and let $\mathfrak{a} \subset k[T_\bullet]$ be the ideal generated by $\mathcal{F}$. Then $V(\mathcal{F}) = V(\mathfrak{a})$, as is easily checked. From this it also follows that
$$V(\mathcal{F}) = V(\mathfrak{a}) = V(f_1, \dots, f_m),$$
whenever $f_1, \dots, f_m$ is a generating system for the ideal $\mathfrak{a}$. By Hilbert's Basis Theorem, every ideal in $k[T_1, \dots, T_n]$ admits a finite generating system. (We say that the polynomial ring in finitely many variables over a field, or more generally over any noetherian ring, is noetherian.) Therefore, every vanishing set $V(F)$ can be written in the form $V(f_1, \dots, f_m)$ for finitely many, suitably chosen polynomials $f_j$.

**Proposition 1.5.** *The sets* $V(\mathcal{F})$, $\mathcal{F} \subset k[T_\bullet]$, *form the closed sets of a topology on* $k^n$, *the* Zariski topology.

*Spelled out explicitly, this means that*

(1) $\emptyset$, $k^n$ *are of this form,*

(2) *finite unions of such sets are again of this form,*

(3) *arbitrary intersections of such sets are of this form.*

*Proof.* (1) We have $\emptyset = V(1)$, $k^n = V(0)$.

(2) By induction, it is enough to consider the union of two closed subsets, say $V(\mathcal{F})$ and $V(\mathcal{G})$. But

$$V(\mathcal{F}) \cup V(\mathcal{G}) = V(fg;\ f \in \mathcal{F},\ g \in \mathcal{G}).$$

In fact, the inclusion $\subseteq$ is clear. For the other inclusion, take a point $t$ in the right hand side which does not lie in $V(\mathcal{F})$. That means $f(t) \neq 0$ for some $f \in \mathcal{F}$. But since $f(t)g(t) = (fg)(t) = 0$ for all $g \in \mathcal{G}$, it follows, that $t \in V(\mathcal{G})$.

(3) For $\mathcal{F}_j \subseteq k[T_1, \dots, T_n]$, $j \in J$, we have

$$\bigcap_{j\in J} V(\mathcal{F}_j) = V\left(\bigcup_{j\in J} \mathcal{F}_j\right).$$

□

**Definition 1.6.** *The topological space $k^n$ with the Zariski topology is denoted by $\mathbb{A}^n(k)$ and called* affine space (over $k$, of dimension $n$)*.*

### (1.4) Bézout's Theorem.

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Next, let us look at *Bézout's theorem*, a relatively elementary, but still non-trivial result in algebraic geometry which at the same time illustrates a typical type of question asked in this theory and several methods that are crucial in (almost) all of algebraic geometry. In particular, it will serve as a motivation for introducing the so-called *projective space*, see Section (1.5).

Let $k$ be a field. For a polynomial $f \in k[X, Y]$, as before we write

$$V(f) = \left\{(x, y) \in k^2;\ f(x, y) = 0\right\},$$

and call this set the *vanishing set of $f$*.

We want to study what we can say, given two such polynomials $f$, $g$, about the set $V(f) \cap V(g)$. More specifically, examples show that typically, this is a finite set, and it is a natural question whether we can determine its cardinality. We start with the following observations:

(1) For a polynomial $p \in k[X]$, $n = \deg(p) > 0$, we have

$$\#\{x \in k;\ p(x) = 0\} \leq n,$$

with equality if $k$ is algebraically closed and if we count each zero $x$ of $p$ with its multiplicity $\operatorname{ord}_x(p) = \max\{r;\ (X - x)^r \mid p\}$.

(2) Let $p \in k[X]$ non-constant and let $f = Y - p(X)$, $g = Y$. We then have a bijection

$$\{x \in k;\ p(x) = 0\} \longleftrightarrow V(f) \cap V(g), \quad x \mapsto (x, 0).$$

Coming back to the general case, let $f, g \in k[X,Y]$. Recall that $k[X,Y]$ is a unique factorization domain. It is easy to see that in case $f$ and $g$ have a common divisor of positive degree, then $V(f) \cap V(g)$ is infinite, at least when $k$ is algebraically closed. Since here we are interested in counting points, we rule out that case, and require that $f$, $g$ are coprime. For a polynomial $f \in k[X,Y]$, we denote by $\deg(f)$ its *total degree*, i.e., for $f = \sum_{i,j} a_{ij} X^i Y^j$, $\deg(f) = \max\{i + j;\ a_{ij} \neq 0\}$.

**Proposition 1.7.** *Let $k$ be a field, and let $f, g \in k[X,Y]$ be coprime, non-constant polynomials. Then*

$$\#(V(f) \cap V(g)) \leq \deg(f) \cdot \deg(g).$$

We will prove this result later, in an improved form. For now, our goal is to discuss this "improved form", by which we mean a refined statement where we actually have equality.

Looking back at the case of a single-variable polynomial $p$ above, it is reasonable to require that $k$ is algebraically closed, and also to expect that we will have to count intersection points with their correct "multiplicity". It is not so hard to write down the definition of multiplicity that will work; we will discuss this in more detail later.

**Definition 1.8.** [Local intersection multiplicity] *Let $k$ be a field, $f, g \in k[X,Y]$, $P = (x,y) \in k^2$ a point. Let $\mathfrak{m} = (X - x, Y - y) \subset k[X,Y]$ (a maximal ideal of the polynomial ring). Then we define*

$$i_P(f,g) = \dim_k k[X,Y]_{\mathfrak{m}}/(f,g),$$

*where $k[X,Y]_{\mathfrak{m}}$ denotes the localization at $\mathfrak{m}$ (i.e., the localization with respect to the multiplicative subset $k[X,Y] \setminus \mathfrak{m}$). (Note that the intersection multiplicity depends on the polynomials $f$, $g$; not just on their vanishing sets.)*

However, looking at the case where $V(f)$ and $V(g)$ are parallel lines in $k^2$ (e.g., $f = Y$, $g = Y - 1$), we see that these changes are not enough in order to obtain equality.

### (1.5) The projective plane $\mathbb{P}^2(k)$.

*Idea.* Add points to $k^2$ so that any two different lines intersect in a point. (While this at first may feel like cheating, it turns out that the resulting construction is extremely useful in algebraic geometry, far beyond Bézout's theorem, also in the sense that it will allow to come back and answer questions that do not mention the newly constructed space.) Setting up the theory will also involve suitably modifying the notion of line; we will come to that later, and then also relate it to lines in $k^2$.

**Definition 1.9.** *Let $k$ be a field. We define the* projective plane $\mathbb{P}^2(k)$ *over $k$, as a set, as*

$$\mathbb{P}^2(k) := \{L \subset k^3 \text{ linear subspace of dimension } 1\},$$

*the set of all lines through the origin in $k^3$.*

Viewing $k^2$ as the affine plane $\{(x, y, 1) \in k^3;\ x, y \in k\}$ in $k^3$, every line through the origin in $k^3$ which is not contained in the $x$-$y$-plane intersects $k^2$ in exactly one point. Thus we obtain an injective map $k^2 \to \mathbb{P}^2(k)$ which we may also write as

$$k^2 \longrightarrow \mathbb{P}^2(k), \quad (x, y) \mapsto \left\langle \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \right\rangle.$$

In this way, we may view $\mathbb{P}^2(k)$ as "$k^2$ with some points added", namely the lines in the $x$-$y$-plane (note that thus for any equivalence class of parallel lines in $k^2$ we have one additional point, and it will turn out that this point "is" (in a sense that we yet must define) the missing intersection point of these parallel lines).

Usually we denote elements of $\mathbb{P}^2(k)$ in terms of their *homogeneous coordinates* which we are going to define next. (That also facilitates, hopefully, to think of elements of $\mathbb{P}^2(k)$, typically, as *points of some space* rather than as lines in some other space, similarly as we think of the elements of $k^2$ as points in the plane.)

For $(x, y, z), (x', y', z') \in k^3 \setminus \{0\}$, define:

$$(x, y, z) \sim (x', y', z') \iff \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \right\rangle$$
$$\iff \exists \lambda \in k^\times : (x', y', z') = \lambda(x, y, z).$$

This is an equivalence relation on $k^3 \setminus \{0\}$. We denote by $(x : y : z)$ the equivalence class of $(x, y, z)$ and obtain a bijection

$$(k^3 \setminus \{0\})/\sim \quad \xrightarrow{1:1} \quad \mathbb{P}^2(k), \qquad (x : y : z) \mapsto \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle.$$

Our next task is to define a suitable notion of *line in the projective plane*. The resulting notion should satisfy (at least) the properties that through any two distinct points, there is a unique line; and that any two distinct lines intersect in a unique point (because our goal was a situation where there are no more "parallel lines"). For the definition, however, a general construction is better suited, namely an analog of the notion of vanishing set of polynomials. However, we have to be careful here, because for an arbitrary polynomial $F \in k[X, Y, Z]$ the value on a point $(x : y : z)$ given in homogeneous coordinates is obviously not well-defined, but will depend on the choice of representative. On the other hand, in order to define vanishing sets, we do not need to compute values, but only need to check whether the outcome is $= 0$ or $\neq 0$. Even this is not possible for general polynomials, but it is possible for the class of *homogeneous* polynomials, which is still large enough to give all that we need. We give the definition in a general form.

**Definition 1.10.** *Let $R$ be a ring. A polynomial $F \in R[X_0, \dots, X_n]$ is called* homogeneous of degree $d$*, if it can be written as a (finite) linear combination of monomials of degree $d$, i.e., in the form*

$$F = \sum_{i_0,\dots,i_n} a_{i_0,\dots,i_n} X_0^{i_0} \cdots X_n^{i_n}$$

*with $a_{i_0,\dots,i_n} \in R$ and $a_{i_0,\dots,i_n} = 0$ whenever $i_0 + \dots + i_n \neq d$.*

**Lemma 1.11.** *Let $R$ be a ring, and let $F \in R[X_0, \dots, X_n]$ be homogeneous of degree $d$. Then for all $\lambda, x_0, \dots, x_n \in R$, we have*

$$F(\lambda x_0, \dots, \lambda x_n) = \lambda^d F(x_0, \dots, x_n).$$

*If $R$ is an infinite field, then the converse is true, as well.*

*Proof.* The first statement is clear. The second one follows (how?) from the fact that over an infinite field, the zero polynomial is the only polynomial in $n + 2$ variables which vanishes at every point of $k^{n+2}$. □

Therefore we may define the vanishing set of a homogeneous polynomial, and more generally, the common vanishing set of a family of homogeneous polynomials (possibly of different degrees). We will look at several explicit examples soon.

**Definition 1.12.** *Let $\mathcal{F} \subseteq k[X, Y, Z]$ be a set of homogeneous polynomials. We define the* vanishing set *as*

$$V_+(\mathcal{F}) = \{(x : y : z);\ F(x, y, z) = 0 \text{ for all } F \in \mathcal{F}\} \quad \subseteq \mathbb{P}^2(k).$$

Similarly as for $k^n$, one proves the following. Oct. 21, 2025

**Proposition 1.13.** *The sets of the form $V_+(\mathcal{F})$, $\mathcal{F} \subseteq k[X, Y, Z]$ a set of homogeneous polynomials, form the closed sets of a topology on $\mathbb{P}^2(k)$, the so-called* Zariski topology.

*Lines in $\mathbb{P}^2(k)$.* We can now define the notion of line in the projective plane and conclude this section by stating the final form of Bézout't theorem.

**Definition 1.14.** *Let $k$ be a field. A* line *in $\mathbb{P}^2(k)$ is a subset of the form $V_+(F)$ for a non-zero homogeneous polynomial of degree* 1.

Explicitly, $F$ as in the definition has the form $aX + bY + cZ$, with $(a, b, c) \neq (0, 0, 0)$. For example $V_+(Z) = \mathbb{P}^2(k) \setminus \iota(k^2)$ (where $\iota\colon k^2 \to \mathbb{P}^2(k)$ is the embedding defined above) is a line. This line is called the *line at infinity* (with respect to our chosen embedding $k^2 \subset \mathbb{P}^2(k)$).

**Proposition 1.15.**
(1) *Let $P_1, P_2 \in \mathbb{P}^2(k)$, $P_1 \neq P_2$. Then there exists $F \in k[X,Y,Z]$ homogeneous of degree 1, $F \neq 0$, such that $P_1, P_2 \in V_+(F)$, and $F$ is uniquely determined up to multiplication by an element $\lambda \in k^\times$.*
(2) *For non-zero linear homogeneous polynomials $F_1, F_2 \in k[X,Y,Z]$, we have*

$$V_+(F_1) = V_+(F_2) \iff \text{there exists } \lambda \in k^\times : F_2 = \lambda F_1.$$

(3) *Let $F_1, F_2 \in k[X,Y,Z]$ be non-zero linear homogeneous polynomials with $V_+(F_1) \neq V_+(F_2)$. Then the set $V_+(F_1) \cap V_+(F_2)$ consists of exactly one element.*

*Proof.* (1) Phrase the problem as a system of linear equations on the coefficients of $F$. We obtain a system with two linearly independent equations and three variables, so the space of solutions is 1-dimensional.

(2) This follows from Part (1) (because any $V_+(F_1)$ contains at least 2 points (more precisely: $\#k + 1$ points)).

(3) Similarly as Part (1) this can be shown by considering a suitable system of linear equations, where the coefficients are given by the coefficients of the equations of $F_1$ and $F_2$, and the variables correspond to the homogeneous coordinates of the point(s) we are looking for in the intersection. □

We can now state the final version of Bézout's theorem. Here, $i_P(F,G)$ is defined similarly as above. (As before, it depends on the actual polynomials $F$, $G$, not just on their vanishing sets.) We will come back to this, and also give a proof of the theorem, later in the course.

**Theorem 1.16.** [Bézout] *Let $F, G \in k[X,Y,Z]$ be non-constant coprime homogeneous polynomials. Then*

$$\sum_{P \in V_+(F) \cap V_+(G)} i_P(F,G) = \deg(F) \cdot \deg(G),$$

*in particular $\#(V_+(F) \cap V_+(G)) \leq \deg(F) \cdot \deg(G)$.*

Similarly to the projective plane, we can analogously define *projective space of dimension $n$ over $k$*,

$$\mathbb{P}^n(k) = (k^{n+1} \setminus \{0\})/\sim,$$

where $(x_0, \dots, x_n) \sim (x'_0, \dots, x'_n)$ if there exists $\lambda \in k^\times$ such that $x'_i = \lambda x_i$ for all $i$.

## (1.6) Homogenization and dehomogenization of polynomials.

Let us look at the relationship between vanishing sets in $k^2$ and in $\mathbb{P}^2(k)$.

**Remark 1.17.** Let $F \in k[X, Y, Z]$ be a homogeneous polynomial. Then $V_+(F) \cap V_+(Z) = V_+(F, Z)$ and thus we can write $V_+(F)$ as the disjoint union

$$V_+(F) = (V_+(F) \cap \iota(k^2)) \sqcup V_+(F, Z).$$

Furthermore, under the identification $k^2 \xrightarrow{1:1} \iota(k^2)$, $V_+(F) \cap \iota(k^2)$ is in bijection with $V(f)$ for $f = F(x, y, 1) \in k[x, y]$. Here $f$ is a polynomial of degree $\leq \deg(F)$, with equality, if $F$ is not divisible by $Z$.

Conversely, given a polynomial $f \in k[x, y]$ we can easily find a homogeneous polynomial such that $f(x, y) = F(X, Y, 1)$ (and hence, by the above remark, $V(f) = V_+(F) \cap \iota(k^2)$, or in other words, $V_+(F)$ consists of $V(f)$ and (possibly) further points lying on the line at infinity $V_+(Z)$).

Namely, we just "fill in powers of $Z$" so as to construct a homogeneous polynomial of degree $\deg(f)$. For example, for $f = y^2 - x^3 + x + 1$, we would take $F = Y^2Z - X^3 + XZ^2 + Z^3$. Generally, given $f = \sum_{i,j} a_{i,j} x^i y^j$ of degree $d$, take $F = \sum_{i,j} a_{i,j} X^i Y^j Z^{d-i-j}$. We call $F$ the dehomogenization (of degree $d$) of $f$.

Note that for $f$ and $F$ related in this way, the polynomial $G = Z \cdot F$ still has the property that $G(x, y, 1) = f(x, y)$, however $V_+(G) = V_+(F) \cup V_+(Z)$, i.e., we get an "unnecessary" (and unwanted) copy of the line at infinity.

**(1.7) More examples.**

Let $k$ be a field, $\mathrm{char}(k) \neq 2$. A vanishing set $V(f) \subset \mathbb{A}^2(k)$ for a polynomial $f$ of degree 3 is called a *cubic curve.* Oct. 22, 2025

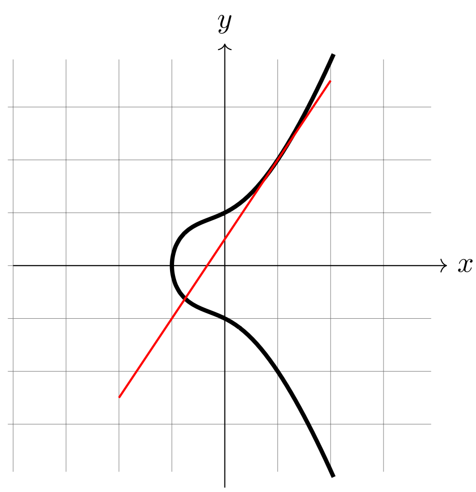
**Example 1.18.** Consider $C = V(f) \subset \mathbb{A}^2(k)$ with

$$f = y^2 - (x + 1)(x^2 + 1).$$

We have

$$\frac{\partial f}{\partial x} = -3x^2 - 2x - 1, \qquad \frac{\partial f}{\partial y} = 2y.$$

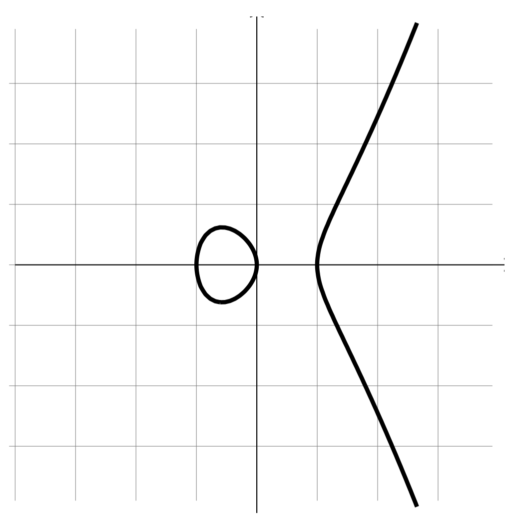
Let $P = (1, 2) \in C$. We have $\frac{\partial f}{\partial x}(P) = -6$, $\frac{\partial f}{\partial y}(P)4$. This implies that over $\mathbb{R}$ (and similarly over $\mathbb{C}$) the function $(x, y) \mapsto f(x, y)$ is approximated well by the linear function $(x, y) \mapsto -6x+4y-2$, and the zero set $V(f)$ is approximated, "in a small neighborhood of $P$" by the zero set of the above linear function, i.e., by the line $V(-6x + 4y - 2)$ (drawn in red).




**Example 1.19.**

Consider $C = V(f) \subset \mathbb{A}^2(k)$ with

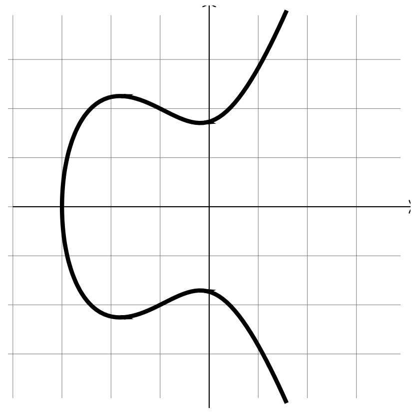
$$f = y^2 - x(x + 1)(x - 1).$$



**Example 1.20.**

Consider $C = V(f) \subset \mathbb{A}^2(k)$ with

$$f = y^2 - (x+3)(x^2+1).$$



**Example 1.21.** Consider $C = V(f) \subset \mathbb{A}^2(k)$ with

$$f = y^2 - x^2(x+1).$$

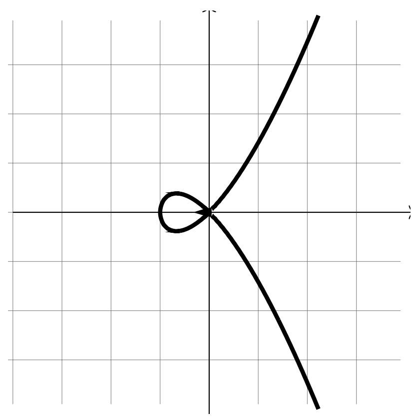
In this case,

$$\frac{\partial f}{\partial x} = -3x^2 - 2x, \quad \frac{\partial f}{\partial y} = 2y$$

and in particular

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0.$$

This corresponds to the fact that there is no well-defined tangent line to $V(f)$ at the point $(0,0)$.

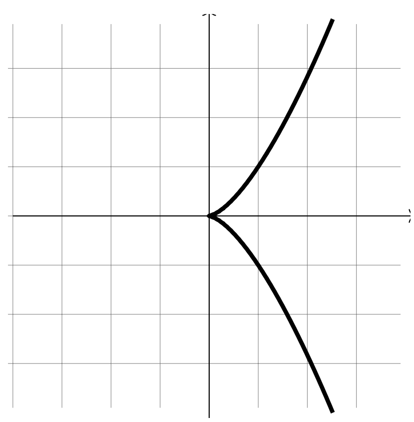


**Example 1.22.**

Consider $C = V(f) \subset \mathbb{A}^2(k)$ with

$$f = y^2 - x^3.$$

In this case once again both partial derivatives of $f$ vanish at $(0,0)$.



**(1.8) Singular and nonsingular points.**

From the examples and the situation over the real and complex numbers, we would like to make the following definition, as one example that we can use some geometric insight, while formally "only manipulating algebraic expressions" (in this case, taking derivatives of polynomials).

**Definition* 1.23.** *Let $k$ be a field and let $f \in k[X,Y]$ be a non-constant polynomial. We say that a point $P = (x_0, y_0) \in V(f)$ is* smooth *(or* non-singular*), if $\left(\frac{\partial f}{\partial X}(P), \frac{\partial f}{\partial X}(P)\right) \neq (0,0)$, and in this case call the line*

$$V\left(\frac{\partial f}{\partial X}(P)\cdot(X-x_0)+\frac{\partial f}{\partial X}(P)\cdot(Y-y_0)\right)$$

*the* tangent line *to $V(f)$ at $P$. Otherwise we call $P$ a* singular point *of $V(f)$.*

**Remark 1.24. This definition does not really make sense!** (that's why I put a *) – more precisely, the property of being a smooth point *depends on the polynomial* $f$, not just on the subset $V(f) \subset k^2$. For example, $V(X) = V(X^2)$, and using the partial derivatives of $f = X$, all points are smooth, but using $f = X^2$ instead, all points are singular. This illustrates that the set $V(f)$ (even if we equip it with the induced topology for the embedding into $k^2$ with the Zariski topology) alone does not carry enough "structure" in order to really do geometry.

For now we will therefore view this as a "definition we would like to make for $V(f)$, but can currently only make after fixing $f$". A little later in the course we will be in a position to fix this problem.

If $k$ is algebraically closed, then there is another option to proceed. (The fact that this is option is not viable for general fields is the reason that "classical" algebraic geometry, e.g., as in [GW1] Chapter 1 or [Ha] Chapter I, is done over an algebraically closed base field.)

To formulate this, recall that a ring $R$ is called *reduced*, if it has no non-zero nilpotent elements, i.e., whenever $x^n = 0$ for some $x \in R$, $n \geq 1$, we must have $x = 0$. For a polynomial $f \in k[x,y]$, the quotient is reduced if and only if there does not exist an irreducible polynomial $g \in k[x,y]$ such that $g^2 \mid f$. In other words, in the decomposition of $f$ into irreducible polynomials in the unique factorization domain $k[x,y]$ each irreducible factor occurs only once.

If $f \in k[x,y]$ is a non-constant polynomial and $f = f_1^{i_1} \cdot \dots \cdot f_r^{i_r}$ is a decomposition of $f$ with $f_i$ irreducible and pairwise distinct, then clearly $V(f) = V(f_1 \cdot \dots \cdot f_r)$, i.e., changing the exponents does not change the vanishing set. It is therefore clear that every $V(f)$ can also be written as the vanishing set of a polynomial for which $k[x,y]/(f)$ is reduced.

Over an algebraically closed field, we have the following strong converse: Given $V \subset k^2$ that has the form "vanishing set of one non-constant polynomial", there is a unique (up to multiplication by scalars in $k^\times$) polynomial $f \in k[x,y]$ such that $V = V(f)$ and such that the ring $k[x,y]/(f)$ is *reduced* (i.e., it has no non-trivial nilpotent elements). When we use this $f$, we get the “right” notion of smooth points. (In fact, it is not difficult to show that for $f$ such that $k[x,y]/(f)$ is *not* reduced, all points are non-smooth in the sense of the above definition applied to $f$.)

In fact, there is the following more general version of this statement. For an ideal $\mathfrak{a} \subset k[T_1,\dots,T_n]$ we denote by

$$\sqrt{a} = \{x \in k[T_\bullet];\ x^n \in \mathfrak{a} \text{ for some } n \geq 0\}$$

its radical. (With notation as above, $\sqrt{(f)} = (f_1 \cdot \dots \cdot f_r)$.) It is easy to see that $k[T_1,\dots,T_n]/\mathfrak{a}$ is reduced if and only if $\mathfrak{a} = \sqrt{\mathfrak{a}}$, and that $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$. Furthermore, we have:

**Theorem 1.25.** *Let $k$ be an algebraically closed field, and let $\mathfrak{a}, \mathfrak{b} \subseteq k[T_1,\dots,T_n]$ be ideals. Then*

$$V(\mathfrak{a}) = V(\mathfrak{b}) \quad \iff \quad \sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}.$$

This is (one version of) Hilbert’s Nullstellensatz. The implication $\Leftarrow$ is easy, as indicated above, and does not require the assumption that $k$ is algebraically closed. The other implication is non-trivial already in the case that $\mathfrak{a} = (1)$, so $V(\mathfrak{a}) = \emptyset$. In this case the statement is equivalent to saying that any family $f_1,\dots,f_r$ of polynomials that does not generate the unit ideal has a common zero, whence the name *Nullstellensatz* (*Nullstelle* is German for *zero (of a polynomial)*).

We will take up this discussion again, and in more detail, later.

**Remark 1.26.** Another perspective on the situation over the real numbers (and similarly over the complex numbers) is the Theorem on inverse functions. It implies, if $P$ is a smooth point of $V(f)$ in the sense of the above definition, that locally (in the analytic, “usual”, topology) around $P$ the set $V(f)$ is diffeomorphic to an open interval in $\mathbb{R}$, i.e., there exists an open $U \subset V(f)$, $P \in U$, and an open interval $V \subset \mathbb{R}$, and bijective differentiable functions $U \to V$ and $V \to U$ that are inverse to each other.

More generally, there is a version for vanishing sets (or more generally, level sets) of continuously differentiable maps $\mathbb{R}^n \to \mathbb{R}$, and even more generally for fibers of continuously differentiable maps $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto (f_1(x),\dots,f_m(x))$, such that the Jacobi matrix (at some point $P$),

$$\left(\frac{\partial f_j}{\partial x_i}(P)\right)_{i,j}$$

has rank $m$. Then locally around $P$, the fiber over $f(P)$ is a differentiable manifold, i.e., is diffeomorphic to an open of $\mathbb{R}^{n-m}$.

See Inverse function theorem (Wikipedia)[1], in particular the section Giving a manifold structure.

[1] https://en.wikipedia.org/wiki/Inverse_function_theorem

For the projective plane, we make the following analogous definition (which again depends on the polynomial $F$, not only on the vanishing set, cf. Remark 1.24

**Definition 1.27.** *Let $k$ be a field and let $F \in k[X,Y,Z]$ be a non-constant homogeneous polynomial. We call a point $P \in V_+(F)$ a* smooth *(or* non-singular*) point of $V_+(F)$, if*

$$\left(\frac{\partial F}{\partial X}(P), \frac{\partial F}{\partial Y}(P), \frac{\partial F}{\partial Z}(P)\right) \neq (0,0,0),$$

*and in this case call the line*

$$V_+\left(\frac{\partial F}{\partial X}(P)\cdot X + \frac{\partial F}{\partial Y}(P)\cdot Y + \frac{\partial F}{\partial Z}(P)\cdot Z\right)$$

*the* tangent line *to $V_+(F)$ at $P$. Otherwise, we call $P$ a* singular *point.*

For the following remarks, the next lemma will be useful; we record it here in the general case of $n+1$ variables. Also note that for a homogeneous polynomial of degree $d$, all partial derivatives are homogeneous of degree $d-1$.

**Lemma 1.28.** (Euler identity) *Let $F \in k[X_0,\dots,X_n]$ be a homogeneous polynomial of degree $d$. Then*

$$\frac{\partial F}{\partial X_0}X_0 + \dots + \frac{\partial F}{\partial X_n}X_n = dF.$$

*Proof.* Since both sides are $k$-linear in $F$, it is enough to check this in case $F = X_0^{\nu_0}\cdots X_n^{\nu_n}$ is a monomial. But then $\frac{\partial F}{\partial X_i}X_i = \nu_i F$ and the stated identity follows immediately. □

**Remark 1.29.**

(1) Euler's identity shows that the tangent line to a smooth point of $V_+(F)$ contains the point $P$.

(2) The two definitions of smooth point are related as follows. Let $F \in k[X,Y,Z]$ be a homogeneous polynomial, $f = F(x,y,1)$, so that $V_+(F) \cap \iota(k^2)$ may be identified with $V(f) \subset k^2$. Cf. Section (1.6). We assume that $f$ is non-constant, and take $P \in V(f)$, say $P = (x_0, y_0)$, and $\iota(P) = (x_0 : y_0 : 1)$.

Then

$$\frac{\partial F}{\partial X}(x,y,1) = \frac{\partial f}{\partial x}, \quad \frac{\partial F}{\partial Y}(x,y,1) = \frac{\partial f}{\partial y}, \tag{1.8.1}$$

as is easily checked, and in particular

$$\frac{\partial F}{\partial X}(x_0,y_0,1) = \frac{\partial f}{\partial x}(P), \quad \frac{\partial F}{\partial Y}(x_0,y_0,1) = \frac{\partial f}{\partial y}(P).$$

This already shows that if $P \in V(f)$ is smooth (with respect to the polynomial $f$, that is), then $\iota(P)$ is a smooth point of $V_+(F)$ (i.e., for

$F$). To show the equivalence, assume that $\iota(P) \in V_+(F)$ such that the partial derivatives of $F$ with respect to $X$ and to $Y$ both vanish. Then Euler's identity shows, since $F(\iota(P)) = 0$, that the partial derivative with respect to $Z$ vanishes, as well, so $\iota(P)$ is not smooth.

Finally, for a smooth point with tangent line $V_+(L)$ to $V_+(F)$ at $\iota(P)$,

$$L = \frac{\partial F}{\partial X}(P) \cdot X + \frac{\partial F}{\partial Y}(P) \cdot Y + \frac{\partial F}{\partial Z}(P) \cdot Z,$$

equation (1.8.1) shows that $V(L(x, y, 1))$ is the tangent to $V(f)$ at $P$. In this sense, the two definitions are compatible.

**(1.9) Smoothness for Cubic Curves.**

Let us understand the notion of smoothness in the special case of cubic curves (compare the earlier examples), more precisely for $V(f)$ with $f$ of the form

$$f = y^2 - (x^3 + ax^2 + bx + c) = y^2 - g(x).$$

As before, assume $\mathrm{char}(k) \neq 2$.

Then

$$\frac{\partial f}{\partial x} = -g'(x), \qquad \frac{\partial f}{\partial y} = 2y.$$

(This shows why the situation is different in characteristic 2, namely then the partial derivative with respect to $y$ vanishes for all points.)

The points $(x_0, y_0) \in V(f)$ where both partial derivatives vanish satisfy

$$y_0 = 0, \quad g(x_0) = g'(x_0) = 0,$$

i.e. $x_0$ is a multiple root of $g$.

**Proposition 1.30.** *For $f$ as above, $V(f)$ is smooth if and only if $g$ is separable (i.e. $g$ has no multiple roots in an algebraic closure $\overline{k}$).*

We may homogenize $f$ to obtain

$$F(X, Y, Z) = Y^2 Z - X^3 - aX^2 Z - bXZ^2 - cZ^3.$$

homogeneous of degree 3. By Remark 1.29, $P \in V(f)$ is smooth if and only $\iota(P) \in V_+(F)$ is smooth, where as usual $\iota$ denotes the embedding $k^2 \to \mathbb{P}^2(k)$. Let us check smoothness at those points of $V_+(F)$ that lie on the line at infinity, i.e., points of the form $(x_0 : y_0 : 1) \in V_+(F)$. Then the vanishing of $F$ amounts to $x_0 = 0$, and since this excludes the possibility of $y_0$ vanishing as well, and we can scale the homogeneous coordinates, we see that $V_+(F) \cap V_+(Z)$ consists of the one point $(0 : 1 : 0)$.

At this point, the partial derivative $\dfrac{\partial F}{\partial Z} = Y^2 - aX^2 - 2bXZ - 3cZ^2$ does not vanish, so it is a smooth point, independently of the choice of $a, b, c$. Therefore, for this special form of $f$ and $F$, $V_+(F)$ is smooth if and only $V(f)$ is smooth, if and only if $g$ is separable.

Oct. 28, 2025

**(1.10) Elliptic Curves and the Group Law.**

**Definition* 1.31.** *Let $k$ be a field. An* elliptic curve over $k$ *is given by a homogeneous polynomial $F$ of degree 3 such that the vanishing set $V_+(F)$ is smooth ("for $F$", at all points of $V_+(F)(\overline{k})$, for an algebraic closure $\overline{k}$ of $k$), together with a fixed point $\mathcal{O} \in E$.*

Typical examples are the curves defined by homogenizations of polynomials of the form

$$y^2 - g(x), \quad g \in k[x] \text{ a separable polynomial of degree 3}$$

that we have studied above. In this case, we can (and typically do) choose the unique point $(0:1:0)$ of $V_+(F)$ on the line at infinity as the distinguished point $\mathcal{O}$.

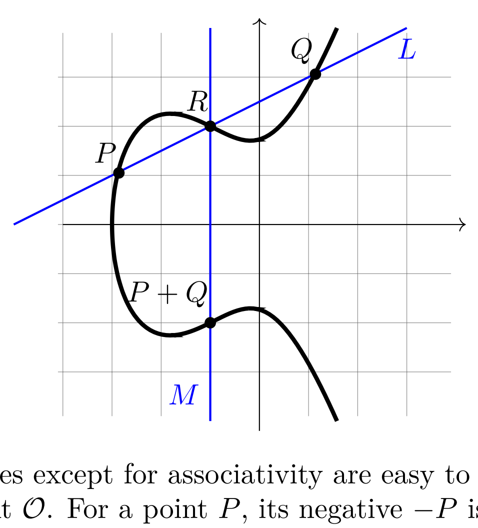
These elliptic curves have an extremely surprising additional structure, as shown by the next proposition. We will assume that $k$ is algebraically closed, so that we can use Bézout's theorem; but see the following remark.

**Proposition 1.32.** *Let $k$ be algebraically closed. Let $E = V_+(F) \subset \mathbb{P}^2(k)$ be smooth with $\deg F = 3$, and let $\mathcal{O} \in E$ be a fixed point. For $P, Q \in E$, let $L \subset \mathbb{P}^2(k)$ be the line through $P, Q$ (or, in case $P = Q$, the tangent to $V_+(F)$ at $P = Q$).*

*Then, counting with multiplicities, the intersection $E \cap L$ has three elements, among them $P$ and $Q$; we express this, and give these points names, by saying that "$E \cap L = \{\{P, Q, R\}\}$ as a multiset". Let $M$ be the line through $\mathcal{O}$ and $R$ (or, in case of equality, the tangent to $V_+(F)$ at this point), write $E \cap M = \{\{\mathcal{O}, R, S\}\}$ and define*

$$P + Q = S.$$

*Then $(E, +)$ is a commutative group with neutral element $\mathcal{O}$.*




*Proof.* All properties except for associativity are easy to check. The neutral element is the point $\mathcal{O}$. For a point $P$, its negative $-P$ is the third point in the intersection of $V_+(F)$ and the line through $P$ and $\mathcal{O}$. The associativity can, in principle, be checked by "direct computation" (write out equations for all the lines involved in terms of coordinates of the points for which one wants to check associativity), but this leads to long, complicated and tedious calculations which are not at all enlightening. For a better, but still elementary proof, see, e.g., [Kn] Section III.3 for a complete proof; cf. also the discussion in [ST] 1.2.

(We will later, but rather next term than this term, be able to give a more enlightening proof based on the Theorem of Riemann–Roch.) □

**Remark 1.33.** One can check that the proposition is still true without the assumption that $k$ be algebraically closed. In fact, for any field extension $k'/k$, we obtain compatible group structures, i.e., $V_+(F)(k) \subseteq V_+(F)(k')$ is a subgroup. The key reason is that whenever a cubic polynomial in one variable over a field $k$ has 2 zeros in $k$, then the third zero also lies in $k$ (always counting zeros with multiplicity). To prove the statement, one way to proceed is to write down formulas for the coordinates of $P+Q$ in terms of the coordinates of $P$ and of $Q$ and see that whenever $P$, $Q$ (and $\mathcal{O}$) have coordinates in $k$, then so does $P+Q$ and $-P$. See [ST] 1.4 or [Kn] III.4.

## Outlook: Advanced results and some open conjectures.

Oct. 29, 2025

### (1.11) The Mordell Conjecture (Faltings' Theorem).

From a number theoretic view, it is an interesting question to determine the number of points of a vanishing set $V_+(F) \subset \mathbb{P}^2(k)$ when $k$ is a *number field*, i.e., a finite extension of $\mathbb{Q}$. For example, if $F$ is linear, then $V_+(F)$ evidently has infinitely many points (whenever $k$ is any infinite field; for a finite field of cardinality $q$, it has $q+1$ points). For $F$ homogeneous of degree 2 the situation is still relatively easy to understand (but we skip this here). However for $F$ of degree $\geq 3$, this is an extremely difficult question, and although a lot of progress has been made over the last 50 years, there are many questions that are still open. We first mention the Theorem of Mordell and Weil that dates back even further and gives some important information in the case of homogeneous cubic polynomials which define a smooth curve, i.e., an elliptic curve.

**Theorem 1.34.** [Mordell–Weil, Mordell 1922 für $K=\mathbb{Q}$; Weil 1928] *Let $K/\mathbb{Q}$ be a finite field extension and let $E$ be an elliptic curve over $K$. Then the abelian group $E(K)$ is finitely generated.*

Depending on the choice of polynomial $F$, the group might be finite or infinite. By the general theory of finitely generated abelian group, we can find a group isomorphism $E(K) \cong \mathbb{Z}^r \times T$ for a finite group $T$ and some natural number $r \geq 0$, called the *rank* of $E$. Even in the case $K = \mathbb{Q}$, there are many open problems around the rank. For example, it is not known whether elliptic curves over $\mathbb{Q}$ of arbitrarily high rank exist. At the time of writing, the best result in this direction is by Elkies and Klagbrun who found (in 2024) an elliptic curve of rank $\geq 29$. The Conjecture of Birch and Swinnerton-Dyer relates the rank of an elliptic curve to a natural number defined in analytic terms (the vanishing order of a certain holomorphic function, the so-called L-function of the elliptic curve).

For a proof of the theorem, see [ST] Chapter 3 (for $K = \mathbb{Q}$), or [Si] Chapter VIII.

For polynomials of higher degree Mordell conjectured that there are only finitely many solutions with coordinates in a fixed number field. This conjecture was proved in 1983 by Faltings, and he received the Fields medal in 1986 in recognition for this proof. We state the result in a slightly more general form (which you can ignore for now, and just read it in the case of the specific example of vanishing sets $V_+(F)$ in $\mathbb{P}^2(k)$).

**Theorem 1.35.** [Mordell Conjecture = Faltings's Theorem] *Let $K/\mathbb{Q}$ be a finite field extension and $C/K$ a smooth projective curve of genus $g \geq 2$, e.g., $C = V_+(F) \subset \mathbb{P}^2(k)$ with $F$ homogeneous of degree $\geq 4$ such that $V_+(F)$ is smooth.*

*Then $C(K)$ is a finite set.*

### (1.12) Fermat's Last Theorem and Modular Curves.

**Theorem 1.36.** ["Fermat's last theorem"; Wiles] *Let $p > 2$ be prime. Then*

$$V_+(X^p + Y^p + Z^p)(\mathbb{Q}) = \{(0:1:-1), (1:0:-1), (1:-1:0)\},$$

*i.e. only the trivial (obvious) solutions exist.*

It follows from Faltings's Theorem that the set on the left hand side is finite whenever $p > 3$, but that theorem does not give any information on the cardinality of this finite set. Wiles's contribution was the following more specific result about elliptic curves over $\mathbb{Q}$.

**Theorem 1.37.** [Taniyama–Shimura–Weil Conjecture; Wiles, 1995] *Every elliptic curve $E/\mathbb{Q}$ is modular.*

Actually, Wiles (together with Taylor) proved a slightly weaker than the theorem stated here; the proof was later completed by Breuil, Conrad, Diamond and Taylor. Ribet, based on an idea of Frey[2], had shown before that this modularity conjecture implies Fermat's Last Theorem. The key idea of Frey was that assuming that $a^p + b^p = c^p$ for $abc \neq 0$, the elliptic curve defined by the (homogenization of the) equation

$$y^2 = x(x - a^p)(y - b^p)$$

[2]Gerhard Frey was a professor at the University of Duisburg-Essen from 1990 to 2009.

has “strange” properties and is seemingly not modular; this was then shown by Ribet.

**Remark 1.38.** We do not explain here what *modular* means. Roughly, it asserts a strong relation between the elliptic curve and a certain “modular form”.

For example, if $E$ is given by $y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Z}$, then modularity implies a precise regularity for the numbers of points

$$\#\{(x,y) \in \mathbb{F}_q^2 \mid y^2 = x^3 + Ax + B\},$$

for each prime power $q$.

### (1.13) The abc Conjecture.

We finish this chapter by a brief discussion of another famous conjecture which at first sight does not have much to do with algebraic geometry (but in fact, it does: for instance, it is equivalent to a conjecture by Szpiro on elliptic curves over $\mathbb{Q}$; indeed, Masser and Oesterlé made their conjecture after studying Szpiro's conjecture and its consequences).

We define the *radical* of a positive integer $n$ as

$$\mathrm{rad}(n) := \prod_{p \text{ prime},\ p|n} p.$$

**Conjecture 1.39.** [abc conjecture, Masser–Oesterlé] *For every $\varepsilon > 0$ there are only finitely many coprime triples $(a, b, c)$ of positive integers with $a+b = c$ and*

$$c > \mathrm{rad}(abc)^{1+\varepsilon}.$$

We also state the following stronger variant, an *explicit form of the abc conjecture.* If $a, b, c \in \mathbb{Z}_{>0}$ are coprime with $a + b = c$, then

$$c \leq \mathrm{rad}(abc)^2.$$

**Example 1.40.** $3+125 = 128 = c$, and $c > 30 = \mathrm{rad}(3\cdot 125\cdot 128)$ illustrates the inequality.

**Remark 1.41.**
(1) It is, somewhat surprisingly, not difficult to prove an analogous statement, where the ring $\mathbb{Z}$ of integers is replaced by the polynomial ring $\mathbb{C}[X]$. See the second problem sheet.
(2) The abc conjecture implies effective versions of the Mordell Conjecture/Faltings's Theorem.

Let us illustrate by showing that the above effective version of the abc conjecture easily implies Fermat's Last Theorem for exponents $n \geq 6$.

In fact, suppose there exist $n \in \mathbb{N}$ and coprime positive integers $x, y, z$ with $x^n + y^n = z^n$. Then

$$z^n \leq \mathrm{rad}(xyz)^2$$

by the abc conjecture, but also

$$\mathrm{rad}(xyz)^2 \le (xyz)^2 < z^6.$$

Putting both inequalities together, we obtain $n < 6$. The cases $n = 3, 4, 5$ are (relatively) easier and have long been known, so the (effective) *abc*-conjecture implies Fermat's Last Theorem.

### (1.14) Problems with Our Approach So Far.

What we have discussed so far was intentionally introductory and not yet systematic. Beyond that, the "theory" so far has some serious problems. Some are easy to fix; others require more serious changes. Desiderata:

- The same vanishing set $V(f)$ (or more generally, $V(f_1, \dots, f_m)$) can be defined by several different polynomials, and the set alone does not "contain enough information" (for example, in order to define smoothness). We would like to equip it with more "geometric structure" which will allow us to not carry around a specific choice of polynomial(s).
- Related to this: A definition of *morphisms* (and hence isomorphisms) between vanishing sets $V(f_1, \dots, f_m)$.
- A more systematic use of *commutative algebra*.
- A theory that works well over *non-algebraically closed fields* (and even over arbitrary commutative rings).
- A more transparent *geometric meaning* of intersection multiplicities $i_P(V_+(F), V_+(G))$ in Bézout's theorem (see earlier sections).

## 2. The prime spectrum of a ring

References: [GW1] (2.1)–(2.4) (or other books that cover scheme theory, e.g., [Mu] Ch. II §1; [Ha] II.2). Nov. 4, 2025

### (2.1) Motivation: Hilbert's Nullstellensatz.

One piece of motivation for the theory we are going to work out is *Hilbert's Nullstellensatz*. To start the discussion, we start with a simple observation.

**Proposition 2.1.** *Let $k$ be a field, $(t_1, \dots, t_n) \in k^n$. Then*

$$(T_1 - t_1, \dots, T_n - t_n) \subset k[T_1, \dots, T_n]$$

*is a maximal ideal. This ideal is the kernel of the evaluation homomorphism*

$$k[T_1, \dots, T_n] \to k, \quad T_i \mapsto t_i,$$

*i.e., a polynomial $f$ lies in the above ideal if and only if $f(t_1, \dots, t_n) = 0$.*

*Proof.* All statements are easy to check. □

**Theorem 2.2.** *Let $k$ be a field, and $\mathfrak{m} \subset k[T_1, \dots, T_n]$ a maximal ideal. Then the field extension $k \subset k[T_1, \dots, T_n]/\mathfrak{m}$ is finite.*

*Proof.* See [GW1] Section (1.3), or [Mu] Ch. I §1, or [Alg2] 4.3[3] for proofs based on Noether Normalization, or, for instance, [AM] Ch. 5, Ch. 7 for (somewhat) different proofs. □

**Corollary 2.3.** *Let $k$ be an algebraically closed field. Then the maximal ideals of the ring $k[T_1, \dots, T_n]$ are precisely the ideals of the form*

$$(T_1 - t_1, \dots, T_n - t_n), \quad (t_1, \dots, t_n) \in k^n,$$

*and we obtain a bijection between $k^n$ and the set of maximal ideals of $k[T_1, \dots, T_n]$.*

*Proof.* Let $\mathfrak{m} \subset k[T_1, \dots, T_n]$ be a maximal ideal. By the theorem, the inclusion $k \to k[T_1, \dots, T_n]/\mathfrak{m}$ is a finite field extension, hence – since $k$ is algebraically closed by assumption – an isomorphism. We define $t_i$ as the image of (the residue class of) $T_i$ under its inverse. Then clearly $(T_1 - t_1, \dots, T_n - t_n) \subset \mathfrak{m}$, and since the left hand side is a maximal ideal, equality follows. □

Under the bijection of the corollary (for $k$ algebraically closed), a vanishing set $V(f_1, \dots, f_m)$ corresponds to the set of all maximal ideals that contain $f_1, \dots, f_m$.

[3] `https://math.ug/a2-ss23/sec-nullstellensatz.html`

**(2.2) The spectrum of a ring.**

Following Grothendieck, and in view of Hilbert's Nullstellensatz, we start a general theory of algebraic geometry by replacing polynomial rings over (algebraically closed) fields by arbitrary ring, and defining for a ring $R$, its prime spectrum

$$\mathrm{Spec}(R) = \{\mathfrak{p} \subset R \text{ prime ideal}\}.$$

(It is better to work with prime ideals than with maximal ideals. One reason is that otherwise the following definition of the map between spectra induced by a ring homomorphism would not work, since in general preimages of maximal ideals under a ring homomorphism are not maximal ideals.)

For an element $f \in R$, we denote by $f(\mathfrak{p})$ the image of $f$ under the ring homomorphism

$$R \longrightarrow R/\mathfrak{p} \longrightarrow \mathrm{Frac}(R/\mathfrak{p}).$$

In particular, we have $f(\mathfrak{p}) = 0$ if and only if $f \in \mathfrak{p}$. (Compare the situation for polynomial rings over fields, and polynomials $f$.)

**Proposition/Definition 2.4.** [Zariski topology on $\mathrm{Spec}(R)$] *Let $R$ be a ring.*

(1) *For a subset $M \subseteq R$, we define the "vanishing set"*

$$V(M) = \{\mathfrak{p} \in \mathrm{Spec}(R);\ M \subseteq \mathfrak{p}\}.$$

*If $\mathfrak{a}$ is the ideal generated by $M$, then $V(M) = V(\mathfrak{a})$. For $M \subseteq M'$, we have $V(M') \subseteq V(M)$. For an element $f \in R$ we also write $V(f)$ for $V(\{f\})$.*

(2) *We have $V(0) = \mathrm{Spec}(R)$, $V(1) = \emptyset$.*

(3) *For a family $\mathfrak{a}_i$ of ideals of $R$, we have*

$$\bigcap_i V(\mathfrak{a}_i) = V\left(\sum_i \mathfrak{a}_i\right).$$

(4) *For ideals $\mathfrak{a}_1, \mathfrak{a}_2 \subseteq R$ we have*

$$V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) = V(\mathfrak{a}_1 \cap \mathfrak{a}_2) = V(\mathfrak{a}_1\mathfrak{a}_2).$$

*In particular, the subsets of* $\mathrm{Spec}(R)$ *of the form $V(\mathfrak{a})$ for ideals $\mathfrak{a} \subseteq R$ form the closed sets of a topology on* $\mathrm{Spec}(R)$*, the so-called* Zariski topology.

*Proof.* Assertions (1), (2) and (3) are easy to check. For (4) note that

$$\mathfrak{a}_1\mathfrak{a}_2 \subseteq \mathfrak{a}_1 \cap \mathfrak{a}_2 \cap \mathfrak{a}_1,$$

and likewise for $\mathfrak{a}_2$, so that we have

$$V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) \subseteq V(\mathfrak{a}_1 \cap \mathfrak{a}_2) \subseteq V(\mathfrak{a}_1\mathfrak{a}_2).$$

Now let $\mathfrak{p} \in V(\mathfrak{a}_1\mathfrak{a}_2)$ and assume that $\mathfrak{p} \notin V(\mathfrak{a}_1)$, say $f \in \mathfrak{a}_1 \setminus \mathfrak{p}$. But then for every $g \in \mathfrak{a}_2$, we have $fg \in \mathfrak{a}_1\mathfrak{a}_2 \subseteq \mathfrak{p}$, and since $\mathfrak{p}$ is a prime ideal and $f \notin \mathfrak{p}$, we get $g \in \mathfrak{p}$. We have shown that $\mathfrak{a}_2 \subseteq \mathfrak{p}$, i.e., that $\mathfrak{p} \in V(\mathfrak{a}_2)$. □

With the notation introduced above, we can also write

$$V(M) = \{\mathfrak{p} \in \operatorname{Spec}(R);\ f(\mathfrak{p}) = 0 \text{ for all } f \in M\}.$$

This in particular shows the analogy with the notation $V(M)$ introduced in Chapter 1. But note that this is only an analogy; we really *redefine* the notation $V(M)$ and from now on will use it only with the new meaning.

We have attached to every ring $R$ a topological space $\operatorname{Spec}(R)$. We extend this definition to a *contravariant functor* from the category of rings to the category of topological spaces, as follows. (This means simply that to each ring homomorphism $\varphi\colon R \to S$ we attach a continuous map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ (note that it goes "in the other direction", whence the term *contravariant*), and this is compatible with composition of homomorphisms, and for the identity map an rings we get the identity map on topological spaces. For more on categories, see, e.g., [Alg2] Section 3.1[4] or [GW1] Appendix A.)

Recall that the inverse image of a prime ideal under a ring homomorphism is a prime ideal.

**Definition 2.5.** *Let $\varphi\colon R \to S$ be a ring homomorphism. We define a map*

$$^a\varphi\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R), \quad \mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q}).$$

*This map is also denoted by* $\operatorname{Spec}(\varphi)$*.*

It is immediate that this construction is compatible with composition of homomorphisms, and for the identity map an rings we get the identity map on topological spaces. Continuity is also easy to check:

**Lemma 2.6.** *Let $\varphi\colon R \to S$ be a ring homomorphism. Then the map $^a\varphi$ is continuous. More precisely, for every ideal $\mathfrak{a} \subseteq R$, we have*

$$(^a\varphi)^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a})).$$

*Proof.* The second statement implies that for every closed subset of $\operatorname{Spec}(R)$ the inverse image under $^a\varphi$ is again closed, and hence that the map $^a\varphi$ is continuous. To prove it, note that for $\mathfrak{q} \in \operatorname{Spec}(S)$ we have

$$\begin{aligned} \mathfrak{q} \in (^a\varphi)^{-1}(V(\mathfrak{a})) &\iff {}^a\varphi(\mathfrak{q}) \in V(\mathfrak{a}) \iff \mathfrak{a} \subseteq \varphi^{-1}(\mathfrak{q}) \\ &\iff \varphi(\mathfrak{a}) \subseteq \mathfrak{q} \iff \mathfrak{q} \in V(\varphi(\mathfrak{a})). \end{aligned}$$

□

**Example 2.7.**
(1) $\operatorname{Spec}(k)$ for $k$ a field,
(2) $\operatorname{Spec}(\mathbb{Z})$,
(3) $\operatorname{Spec}(k \times k)$ for $k$ a field, Nov. 5, 2025
(4) for $R$ a principal ideal domain, the prime ideals are $(0)$ (with closure all of $\operatorname{Spec}(R)$) and the ideals $(f)$ for $f$ irreducible (the latter being maximal ideals and hence corresponding to closed points). This applies in particular to $\operatorname{Spec}(k[T])$ for $k$ a field. If $k$ is assumed to be algebraically closed

[4] `https://math.ug/a2-ss23/subsec-kategorien.html`

then the irreducible polynomials are precisely the linear polynomials and $\operatorname{Spec}(k[T]) = \{(X-a);\ a \in k\} \cup \{(0)\}$. For $k=\mathbb{R}$, in addition to the linear polynomials, there are irreducible polynomials of degree 2, as well, namely all polynomials that decompose in $\mathbb{C}[T]$ as $(X-a)(X-\overline{a})$ for some $a \in \mathbb{C}\setminus\mathbb{R}$. The map $\operatorname{Spec}(\mathbb{C}[T]) \to \operatorname{Spec}(\mathbb{R}[T])$ corresponding to the inclusion $\mathbb{R}[T] \subset \mathbb{C}[T]$ maps $(0) \mapsto (0)$ and

$$(X-a) \mapsto (X-a)\ (a\in\mathbb{R}), \qquad (X-a)\mapsto ((X-a)(X-\overline{a}))\ (a\in\mathbb{C}\setminus\mathbb{R}).$$

**Proposition 2.8.** *Let $R$ be a ring and $\mathfrak{a}\subseteq R$ an ideal. Denote by $\pi\colon R\to R/\mathfrak{a}$ the canonical projection. Then the map ${}^a\pi$ induces a homeomorphism (i.e., a bijective continuous map with continuous inverse map)*

$$\operatorname{Spec}(R/\mathfrak{a}) \to V(\mathfrak{a}).$$

*Proof.* Since (prime) ideals in the quotient correspond bijectively to (prime) ideals in $R$ which contain $\mathfrak{a}$, the map $\operatorname{Spec}(R/\mathfrak{a}) \to \operatorname{Spec}(R)$ has image $V(\mathfrak{a})$ and is injective. Since we know that it is continuous, it only remains to show that the inverse map is continuous, as well. In other words, we need to check that ${}^a\pi$ maps closed sets to closed sets. But one checks easily that

$${}^a\pi(V(\overline{\mathfrak{b}}) = V(\mathfrak{a}) \cap V(\pi^{-1}(\overline{\mathfrak{b}}))$$

which is closed in $V(\mathfrak{a})$, as desired. □

Note that it is easy (i.e., you should do it …) to construct examples of a ring $R$ and ideals $\mathfrak{a}\neq\mathfrak{b}$ with $V(\mathfrak{a}) = V(\mathfrak{b})$. The following result clarifies the situation. Recall the notion of *radical* of an ideal; for $\mathfrak{a}\subseteq R$ its radical is

$$\sqrt{\mathfrak{a}} = \{f\in R;\ \exists n\ge 1 : f^n\in\mathfrak{a}\} = \bigcap_{\mathfrak{p}\in V(\mathfrak{a})} \mathfrak{p},$$

an ideal containing $\mathfrak{a}$. For the second equality, see [Alg2] Satz 2.59[5]. We call an ideal $\mathfrak{a}\subseteq R$ a *radical ideal*, if $\sqrt{\mathfrak{a}} = \mathfrak{a}$.

**Proposition 2.9.** *Let $R$ be a ring. For $Y\subseteq \operatorname{Spec}(R)$, write $I(Y) = \bigcap_{\mathfrak{p}\in Y}\mathfrak{p}$.*

*The maps $\mathfrak{a}\mapsto V(\mathfrak{a})$ and $Y\mapsto I(Y)$ satisfy*

(1) $V(I(Y)) = \overline{Y}$,

(2) $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$,

*and in particular induce a bijection between the set of all radical ideals of $R$ and the set of all closed subsets of* $\operatorname{Spec}(R)$.

*Proof.* It is clear that both $V(-)$ and $I(-)$ are inclusion reversing. Furthermore, $I(Y)$, being an intersection of radical ideals, is itself a radical ideal for every $Y$. Since $V(\mathfrak{a})$ is closed for every $\mathfrak{a}$, the final statement follows from (1) and (2).

Let us show that $V(I(Y)) = \overline{Y}$ for every subset $Y\subseteq\operatorname{Spec}(R)$. Clearly the left hand side is closed and contains $Y$, so we have $\supseteq$. To show $\subseteq$ we need

[5] `https://math.ug/a2-ss23/sec-radikale.html#stz-beschr-rad-a`

to show that $V(I(Y))$ is the smallest closed subset containing $Y$, i.e., that whenever $Y \subseteq V(\mathfrak{a})$, then $V(I(Y)) \subseteq V(\mathfrak{a})$. But if $Y \subseteq V(\mathfrak{a})$, then $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in Y$, so $\mathfrak{a} \subseteq I(Y)$, and hence $V(I(Y)) \subset V(\mathfrak{a})$ as desired.

Now we show that $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for every ideal $\mathfrak{a} \subset R$. But the radical of $\mathfrak{a}$ can be described as

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R),\ \mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p},$$

and this is precisely $I(V(\mathfrak{a}))$. □

In particular, for every $\mathfrak{p} \in \operatorname{Spec}(R)$,

$$\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}).$$

From this we see that a point $\mathfrak{p} \in \operatorname{Spec}(R)$ is closed (i.e., $\{\mathfrak{p}\}$ is a closed subset of $\operatorname{Spec}(R)$) if and only if $\mathfrak{p}$ is a maximal ideal.

**Definition 2.10.** *Let $R$ be a ring. For $f \in R$ we define*

$$D(f) = \operatorname{Spec}(R) \setminus V(f)$$

*and call the subsets of* $\operatorname{Spec}(R)$ *of this form* principal open subsets.

With the notation introduced above we may write

$$D(f) = \{\mathfrak{p} \in \operatorname{Spec}(R);\ f(\mathfrak{p}) \neq 0\},$$

and correspondingly we sometimes think of $D(f)$ as the *non-vanishing set of $f$*.

**Proposition 2.11.** *Let $R$ be a ring.*

(1) *The sets $D(f)$ for $f \in R$ form a basis of the topology of* $\operatorname{Spec}(R)$*, i.e., every open subset of* $\operatorname{Spec}(R)$ *can be written as a union of subsets of this form.*
(2) *The family of sets $D(f)$ is stable under taking finite intersections.*

*Proof.* For (1) take $U \subseteq \operatorname{Spec}(R)$ open, say $U = \operatorname{Spec}(R) \setminus V(\mathfrak{a})$. Then $V(\mathfrak{a}) = \bigcap_{f \in \mathfrak{a}} V(f)$, hence $U = \bigcup_{f \in \mathfrak{a}} D(f)$. For (2) note that $D(f) \cap D(g) = D(fg)$ and that the intersection with empty index set, $\operatorname{Spec}(R)$ equals $D(f)$. □

**Proposition 2.12.** *Let $R$ be a ring and let $f \in R$. Denote by $R_f$ the localization of $R$ with respect to $f$. Then the ring homomorphism $\tau\colon R \to R_f$, $x \mapsto \frac{x}{1}$, induces a homeomorphism*

$$\operatorname{Spec}(R_f) \to D(f).$$

*Proof.* The set of prime ideals in a localization $S^{-1}R$ of $R$ is in bijection to the set of prime ideals $\mathfrak{p}$ in $R$ with $\mathfrak{p} \cap S = \emptyset$, via $\mathfrak{P} \mapsto \tau^{-1}(\mathfrak{P})$ and $\mathfrak{p} \mapsto \mathfrak{p}S^{-1}R = \left\{\frac{a}{s};\ a \in \mathfrak{p}, s \in S\right\}$. This implies that the continuous map $\operatorname{Spec}(R_f) \to \operatorname{Spec}(R)$ restricts to a bijective (and, of course, still continuous) map $\operatorname{Spec}(R_f) \to D(f)$. To check that it is a homeomorphism, we need to check that open sets in $\operatorname{Spec}(R_f)$ have opens in $D(f)$ as their image. It is enough to check this for principal open subsets $D(g/f^i)$, because those are a basis of the topology, and such a set has image $D(fg) = D(f) \cap D(g)$ in $\operatorname{Spec}(R)$, which is open in $D(f)$. (Equivalently, one could check that closed sets have closed image in $D(f)$. This is also easy: for an ideal $\mathfrak{a} \subseteq R_f$, the image of $V(\mathfrak{a})$ in $\operatorname{Spec}(R)$ is $D(f) \cap V(\tau^{-1}(\mathfrak{a})) \cap D(f)$ which is closed in $D(f)$.) □

**Proposition 2.13.** *Let $R$ be a ring and $f \in R$. Then $D(f)$ (with the induced topology) is quasi-compact*[6] *(i.e., for every cover $D(f) = \bigcup_{i\in I} U_i$ by open subsets there exists a finite subset $I' \subseteq I$ with $D(f) = \bigcup_{i\in I'} U_i$). In particular,* $\operatorname{Spec}(R) = D(1)$ *is quasi-compact.*

*Proof.* Exercise (Problem Sheet 3). □

Nov. 11, 2025

**(2.3) Generic points.**

Recall from Problem Sheet 1 the notion of *irreducible* topological space. (A topological space $X$ is called *irreducible*, if $X \neq \emptyset$ and $X$ cannot be written as the union of two proper closed subsets, or equivalently, if $X \neq \emptyset$ and any two non-empty open subsets of $X$ have non-empty intersection.)

Clearly, if $X$ is a topological space and $x \in X$, then the closure $\overline{\{x\}}$ is irreducible (because any non-empty open of this set contains $x$). Topological spaces that arise as spectra of ring have a sort of converse property:

**Proposition 2.14.** *Let $R$ be a ring, $X = \operatorname{Spec}(R)$, and $Z \subseteq X$ a closed subset.*

(1) *The set $Z$ (with the induced topology) is irreducible if and only if $I(Z)$ is a prime ideal.*
(2) *If the closed subset $Z$ is irreducible, then there exists a unique $\eta \in Z$ such that $\overline{\{\eta\}} = Z$, and we call $\eta$ the* generic point *of $Z$.*

*Proof.* Let $\mathfrak{a} = I(Z)$, so $\mathfrak{a} \subseteq R$ is a radical ideal with $Z = V(\mathfrak{a})$. If $\mathfrak{a} = \mathfrak{p}$ is a prime ideal, then $Z = V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ as we have seen above, and in particular $Z$ is irreducible with generic point $\mathfrak{p}$. Since for prime ideals, $V(\mathfrak{p}) = V(\mathfrak{q})$ if and only if $\mathfrak{p} = \mathfrak{q}$ (because prime ideals are radical ideals) this argument also shows the uniqueness statement of (2).

Thus it only remains to show that whenever $Z$ is irreducible, then $I(Z)$ is prime. First note that $Z \neq \emptyset$ implies $I(Z) \neq R$. Now for $f, g \in R$ with $fg \in I(Z)$, we have $Z \subseteq V(fg) = V(f) \cup V(g)$, and if $Z$ is irreducible, it follows that $Z \subseteq V(f)$ or $Z \subseteq V(g)$. Without loss of generality, we have $Z \subseteq V(f)$, say. But then $f \in \sqrt{(f)} = I(V(f)) \subseteq I(Z)$. □

[6] The terminology here follows the French/Bourbaki convention, which also became the standard in algebraic geometry, that a *compact* space is quasi-compact in the sense explained here *and Hausdorff* (which $\operatorname{Spec}(R)$ typically is not).

More precisely, one can show that for any subset $Y \subseteq \mathrm{Spec}(R)$ (not necessarily closed), $Y$ is irreducible if and only if $I(Y)$ is a prime ideal. (But $Y$ need not contain a generic point!) This follows from the following lemma.

**Lemma 2.15.** *Let $X$ be a topological space, and $Y \subseteq X$ a subset with closure $\overline{Y}$. Then $Y$ is irreducible if and only if $\overline{Y}$ is irreducible.*

*Proof.* Use that for any open $U \subseteq X$, we have $Y \cap U = \emptyset$ if and only if $\overline{Y} \cap U = \emptyset$. □

## 3. Sheaves

References: [GW1] (2.5)–(2.8) or [Ha] II.1.

### (3.1) The notion of sheaf.

We have attached to every ring $R$ a topological space $\mathrm{Spec}(R)$, but this topological space "retains very little information" about the ring $R$. For example, for every field $k$ we get the same topological space as its prime spectrum. To achieve our goal of "making $\mathrm{Spec}(R)$ into a geometric object", we follow the slogan that *the geometry of a "space" is determined by the functions on the space*, meaning that for each kind of geometry (topology, differential geometry, complex geometry) there is a natural notion of function (continuous, differentiable, holomorphic), and this is a characteristic feature of the whole theory.

For us, the intuition is that elements of the ring $R$ should be viewed as functions on $\mathrm{Spec}(R)$, but since the elements of $R$ are not "really" functions, it is useful to introduce a more abstract framework that allows us to talk about (and gain intuition from) the previously mentioned cases, but which also applies to the prime spectra of rings.

As it turns out, the following properties are crucial for the "functions" we want to consider. Let $X$ be a topological space.

- A function might be defined on all of $X$, or on some smaller open subset of $X$ (the "domain of definition" of the function). We want to allow the functions to have poles at some point of $X$ and therefore do not ask that the domain of definition is always equal to $X$.
- Functions should be determined by "local information" – since we do not want to talk of the values of a function, we will instead talk about restrictions of a function to open subsets within its "domain of definition", and require that it is determined by the restrictions to open subsets that cover the domain of definition, and also that functions can be defined by "gluing with respect to an open cover" (see below).

(We are deliberately vague about the "target" of our "functions". In differential geometry it would be $\mathbb{R}$, in complex geometry it would be $\mathbb{C}$, but in algebraic geometry we do not really have functions and therefore do not really have a target of functions at our disposal.)

**Definition 3.1.** *Let $X$ be a topological space. A* presheaf $\mathscr{F}$ *(of sets) on $X$ is given by the following data:*

(a) *for each open $U \subseteq X$, a set $\mathscr{F}(U)$,*
(b) *for each pair $U \subseteq V \subseteq X$ of open subsets, a map ("restriction map") $\mathrm{res}^V_U \colon \mathscr{F}(V) \to \mathscr{F}(U)$,*

*such that $\mathrm{res}^U_U = \mathrm{id}_{\mathscr{F}(U)}$ for every $U$ and $\mathrm{res}^W_U = \mathrm{res}^V_U \circ \mathrm{res}^W_V$ for all $U \subseteq V \subseteq W$ open.*

**Definition 3.2.** *Let $X$ be a topological space and $\mathscr{F}$, $\mathscr{G}$ be presheaves on $X$. A* morphism $\varphi \colon \mathscr{F} \to \mathscr{G}$ of presheaves *is a family of maps $\varphi_U \colon \mathscr{F}(U) \to \mathscr{G}(U)$ of maps, $U \subseteq X$ open, such that for all pairs $U \subseteq V \subseteq X$ of open subsets, the diagram*

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\varphi_V} & \mathscr{G}(V) \\ \big\downarrow \mathrm{res}^V_U & & \big\downarrow \mathrm{res}^V_U \\ \mathscr{F}(U) & \xrightarrow{\varphi_U} & \mathscr{G}(U) \end{array}$$

*commutes.*

Notation: We often write $s_{|U}$ for $\mathrm{res}^V_U(s)$ ($s \in \mathscr{F}(V)$). The elements of $\mathscr{F}(U)$ are also called *sections* of the presheaf on the open set $U$. One also writes $\Gamma(U, \mathscr{F})$ instead of $\mathscr{F}(U)$.

While the notion of presheaf provides the basic framework to talk about (generalizations of) functions on a topological space, it is much too general and does not capture enough properties that (even generalized) "functions" should have. It turns out that the crucial property is that functions should be determined by their restrictions to an open cover, and that it should be possible to "specify a function locally", i.e., on an open cover, provided that the obvious compatibility condition on intersections is satisfied. This observation is turned into the definition of *sheaf*, as follows.

**Definition 3.3.** *Let $X$ be a topological space. A presheaf $\mathscr{F}$ (of sets) on $X$ is called a* sheaf *(of sets), if the following condition is satisfied. For every open subset $U \subseteq X$ and every cover $U = \bigcup_{i\in I} U_i$ by open subsets of $X$, the diagram*

$$\mathscr{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathscr{F}(U_i) \overset{\sigma}{\underset{\sigma'}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathscr{F}(U_i \cap U_j)$$

$$s \longmapsto (s_{|U_i})_i$$

$$(s_i)_i \overset{\sigma}{\longmapsto} (s_{i|U_i\cap U_j})_{i,j}$$

$$(s_i)_i \underset{\sigma'}{\longmapsto} (s_{j|U_i\cap U_j})_{i,j}$$

*is* exact*, i.e., the map $\rho$ is injective, and the image of $\rho$ is the set of elements $(s_i)_{i\in I}$ such that $\sigma((s_i)_{i\in I}) = \sigma'((s_i)_{i\in I})$.*

*For sheaves $\mathscr{F}$, $\mathscr{G}$ on $X$, a* morphism $\mathscr{F} \to \mathscr{G}$ of sheaves *is a morphism between the presheaves $\mathscr{F}$ and $\mathscr{G}$.*

Pedantic remark: It follows from the definition (applied to $U = \emptyset$, $I = \emptyset$) that for every sheaf $\mathscr{F}$, the set $\mathscr{F}(\emptyset)$ has precisely one element. Nov. 12, 2025

**Examples 3.4.** Typical examples of sheaves (and one non-example) one should have in mind are the following.

(1) Let $X$ be a topological space and $Y$ a set. Setting, for $U \subseteq X$ open, $\mathscr{F}(U) = \mathrm{Map}(X, Y)$ (the set of all maps $U \to Y$) defines a sheaf on $X$.
(2) Let $X$ and $Y$ be topological spaces. Setting $\mathscr{F}(U) = \mathrm{Map}_{\mathrm{cont}}(X, Y)$ (the set of all *continuous* maps $U \to Y$) defines a sheaf on $X$.
(3) Let $X \subseteq \mathbb{R}^n$ open (or any differentiable manifold). Setting $\mathscr{F}(U) = C^\infty(U)$, the set of infinitely often differentiable functions $U \to \mathbb{R}$, defines a sheaf on $X$.
(4) Let $X \subseteq \mathbb{C}^n$ open (or any complex manifold). Setting $\mathscr{F}(U) = \mathrm{Hol}(U)$, the set of holomorphic functions $U \to \mathbb{C}$, defines a sheaf on $X$.
(5) Let $X = \mathbb{R}$ (with the analytic topology). Setting $\mathscr{F}(U) = \{f\colon U \to \mathbb{R} \text{ bounded}\}$ defines a presheaf on $\mathbb{R}$ which is not a sheaf. (Similarly: bounded and continuous; or bounded and differentiable.)

Typically, the restriction maps $\mathrm{res}^V_U$ are neither injective nor surjective. If the map is not surjective, we may think of it as saying that there are "functions" (more precisely, sections of the sheaf) defined on $U$ but which do not extend (because they may "have poles" at points of $V \setminus U$) to the larger set $V$. On the other hand, a "function" on $V$ cannot usually be expected to be determined by its values on a smaller open set $U$; so in the above examples (1), (2), (3) the restriction maps will be injective only in trivial cases. However in complex analysis (Example (4) above) there is the interesting result (the "identity theorem") that the restriction map $\mathrm{res}^V_U$ is injective whenever $\emptyset \neq U \subseteq V$ and $V$ is connected.

It follows easily from the sheaf axioms that for $U = U_1 \cap U_2$ with $U_1$, $U_2$ open and $U_1 \cap U_2 = \emptyset$, the natural map $\mathscr{F}(U) \to \mathscr{F}(U_1) \times \mathscr{F}(U_2)$ induced by the restriction maps is an isomorphism.

Often the sets of sections carry more structure, for example, in the above examples of sheaves of actual functions (with certain properties such as continuity or differentiability) with target a ring, we can actually add and multiply functions by using the addition and multiplication on the target, so that the sets $\mathscr{F}(U)$ in this case naturally carry a ring structure and the restriction maps are ring homomorphism. This is of course a useful piece of information to remember, and we therefore make the following definition.

**Definition 3.5.** *If $X$ is a topological space and $\mathscr{F}$ a (pre-)sheaf on $X$, and all $\mathscr{F}(U)$ are equipped with the structure of group / abelian group / ring / module over a (fixed) ring $R$ / ..., and all restriction maps are homomorphisms* *for the respective structure, then we speak of a* (pre-)sheaf of groups / abelian groups / rings / $R$-modules / ….

Often the following construction is useful.

**Definition 3.6.** [Restriction of (pre-)sheaf to open subset] *Let $\mathscr{F}$ be a presheaf on a topological space $X$, and let $U \subseteq X$ be an open subset. Then the restriction of $\mathscr{F}$ to $U$ is the presheaf $\mathscr{F}_{|U}$ on $U$ given by $\mathscr{F}_{|U}(V) = \mathscr{F}(V)$ for all $V \subseteq U$ open, and the same restriction maps as those for $\mathscr{F}$. If $\mathscr{F}$ is a sheaf, then so is $\mathscr{F}_{|U}$.*

Our next goal is to define a sheaf of rings on $X = \operatorname{Spec}(R)$ for a ring $R$. This sheaf will be denoted $\mathscr{O}_X$ and called the *structure sheaf* on $\operatorname{Spec}(R)$. The underlying idea is the following. We said that we want to view elements of $R$ as (a kind of) functions on $X$, so we will start by setting $\mathscr{O}_X(X) = R$. For a general open subset $U \subset X$ it is however not so clear what to do. However, for principal opens $D(f) \subseteq X$, there is a natural candidate. Namely, we have seen that there is a natural homeomorphism $\operatorname{Spec}(R_f) \cong D(f)$, and since we have already made a guess what the ring of functions on the left hand side should be (namely $R_f$), we will set $\mathscr{O}_X(D(f)) = R_f$. (We will check later that this is well-defined, i.e., that whenever $D(f) = D(g)$, we have a canonical identification $R_f = R_g$.) At this point we can already (and will, see below) check that this definition satisfies the conditions in the definition of sheaves (i.e., the "gluing of sections"); the computation is not so difficult, but this is a crucial point of the theory. Having checked this, philosophically, we can expect that this should be enough information in order to define $\mathscr{O}_X$, because the $D(f)$ form a basis of the topology, and a sheaf should be determined by local information. This is in fact a general result on sheaves, and we will prove it below.

### (3.2) Sheaves on a basis of the topology.

In this section we fix a topological space $X$ and a basis $\mathcal{B}$ of the topology of $X$ (recall that this means that $\mathcal{B}$ is a set of open subsets of $X$ such that every open subset of $\mathcal{B}$ can be written as a union of elements of $\mathcal{B}$). Things simplify if $\mathcal{B}$ satisfies in addition the property that any finite intersections of open subsets lying in $\mathcal{B}$ is again an element of $\mathcal{B}$. This is satisfied for the basis of principal open subsets of the Zariski topology of the spectrum of a ring, the situation relevant for us, so the reader is advised to make this extra assumption.

**Definition 3.7.**

(1) *A* presheaf $\mathscr{F}$ on the basis $\mathcal{B}$ of the topology *is given by a set $\mathscr{F}(U)$ for every $U \in \mathcal{B}$ and a restriction map* $\operatorname{res}^V_U \colon \mathscr{F}(V) \to \mathscr{F}(U)$ *for every pair of open subsets $U, V \in \mathcal{B}$ with $U \subseteq V$, such that* $\operatorname{res}^U_U = \operatorname{id}_{\mathscr{F}(U)}$ *for every $U \in \mathcal{B}$ and* $\operatorname{res}^W_U = \operatorname{res}^V_U \circ \operatorname{res}^W_V$ *for all open subsets $U, V, W \in \mathcal{B}$, $U \subseteq V \subseteq W$.*

(2) *A presheaf $\mathscr{F}$ on $\mathcal{B}$ is called a* sheaf on $\mathcal{B}$, *if for every $U \in \mathcal{B}$, every cover $U = \bigcup_i U_i$ with $U_i \in \mathcal{B}$ and every open cover $U_i \cap U_j = \bigcup_k U_{ijk}$ with $U_{ijk} \in \mathcal{B}$, the sequence*

$$\mathscr{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathscr{F}(U_i) \underset{\sigma'}{\overset{\sigma}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \prod_k \mathscr{F}(U_{ijk})$$

$$s \longmapsto (s_{|U_i})_i$$

$$(s_i)_i \overset{\sigma}{\longmapsto} (s_{i|U_{ijk}})_{i,j}$$

$$(s_i)_i \underset{\sigma'}{\longmapsto} (s_{j|U_{ijk}})_{i,j}$$

*is exact.*

Equivalently, in (2) it suffices to ask the exactness for every open cover $U = \bigcup_i U_i$, but to fix one open cover $U_i \cap U_j = \bigcup_k U_{ijk}$ for each pair $i, j$, rather than check it for all such covers. In particular, if $\mathcal{B}$ is stable under finite intersections, then one can just cover $U_i \cap U_j$ "by itself", so that one can use "the same sequence as in the definition of a sheaf".

**Definition 3.8.** *Let $\mathscr{F}$, $\mathscr{G}$ be presheaves on $\mathcal{B}$. A morphism $f\colon \mathscr{F} \to \mathscr{G}$ is given by a collection of maps $f_U\colon \mathscr{F}(U) \to \mathscr{G}(U)$ for all $U \in \mathcal{B}$, such that for all $U \subseteq V$, $U, V \in \mathcal{B}$, we have $\mathrm{res}^V_U \circ f_V = f_U \circ \mathrm{res}^V_U$ (where on the left we use the restriction map for $\mathscr{G}$, on the left hand side that for $\mathscr{F}$).*

*For sheaves $\mathscr{F}$, $\mathscr{G}$ a morphism $\mathscr{F} \to \mathscr{G}$ of sheaves on $\mathcal{B}$ is a morphism of the underlying presheaves.*

Together with the obvious identity morphisms and composition of morphisms we obtain the categories of presheaves on $\mathcal{B}$ and of sheaves on $\mathcal{B}$.

It is clear that we can restrict sheaves (and morphisms) on $X$ to $\mathcal{B}$.

**Proposition 3.9.** *For every sheaf $\mathscr{F}$ on $X$ (in the sense of Definition 3.3), the restriction $\mathscr{F}_{|\mathcal{B}}$ given by $\mathscr{F}_{|\mathcal{B}}(U) = \mathscr{F}(U)$ for all $U \in \mathcal{B}$, and similarly for the restriction maps, is a sheaf on $\mathcal{B}$.*

*Similarly any morphism $f\colon \mathscr{F} \to \mathscr{G}$ of sheaves on $X$ induces by restriction a morphism $f_{|\mathcal{B}}\colon \mathscr{F}_{|\mathcal{B}} \to \mathscr{G}_{|\mathcal{B}}$.*

For sheaves, it is reasonable to expect that we can also go in the other direction, i.e., recover a sheaf from its values (including the restriction maps) on $\mathcal{B}$, or more generally, given any sheaf $\mathscr{F}'$ on $\mathcal{B}$ construct a sheaf $\mathscr{F}$ on $X$ such that $\mathscr{F}_{|\mathcal{B}}$ is $\mathscr{F}'$. Furthermore this construction should also be compatible with morphisms. (Of course, a similar result cannot hold true for arbitrary presheaves.) Nov. 18, 2025

The restriction to $\mathcal{B}$ and extension to all open subsets of $X$ are inverse to each other – but only if this is formulated in the right way. It is impossible to achieve that the extension of $\mathscr{F}_{|\mathcal{B}}$ is *equal* to $\mathscr{F}$; rather the best one can hope for is that it is *isomorphic* to $\mathscr{F}$, and that these isomorphisms are compatible with morphisms of sheaves. This situation is best captured by the notion of *equivalence of categories*, see the next section for a short discussion.

**Proposition 3.10.** *The restriction functor $\mathscr{F} \mapsto \mathscr{F}_{|\mathcal{B}}$ from the category of sheaves on $X$ to the category of sheaves on $\mathcal{B}$ is an equivalence of categories.*

*Sketch of proof.* The key point is the construction of a sheaf $\mathscr{F}$ on $X$ given a sheaf $\mathscr{F}'$ on $\mathcal{B}$. The idea of defining $\mathscr{F}(U)$ for an arbitrary open $U \subseteq X$ is easy to explain. Assume that we already had constructed the sheaf $\mathscr{F}$. Then for every cover $U = \bigcup U_i$ with $U_i \in \mathcal{B}$, we could recover $\mathscr{F}(U)$ from the sheaf sequence as a subset of $\prod_i \mathscr{F}(U_i) = \prod_i \mathscr{F}'(U_i)$. If $\mathcal{B}$ is stable under finite intersections, then the intersections $U_i \cap U_j$ are also in $\mathcal{B}$ and we can directly express the compatibility condition cutting our $\mathscr{F}(U)$ inside this products in terms of $\mathscr{F}'$. In general, one can proceed similarly as in the definition of the notion of sheaf on $\mathcal{B}$.

In order to check that this construction has the correct properties, it is however slightly inconvenient that it depends on the choice of a cover of $U$. This problem may be circumvented by simply using the cover of $U$ given by *all* elements of $\mathcal{B}$ that are contained in $U$. This has the additional advantage that all intersections arising in the definition of the sheaf axioms are covered by open subsets that themselves occur in the cover, whence it is enough to simply ask for the compatibility with all restrictions, in the following sense: We define

$$\mathscr{F}(U) = \Big\{(s_V)_V \in \prod_{V\in\mathcal{B},\ V\subseteq U} \mathscr{F}'(V);\ s_{V|W} = s_W \text{ for all } W \subseteq V \subseteq U,\ V, W \in \mathcal{B}\Big\}.$$

Similarly as in the first paragraph, the sheaf axioms imply that this is the only possible candidate for $\mathscr{F}(U)$. It is then not difficult to define restriction maps, to define the extension of morphisms of sheaves, and to show that this extension functor is a quasi-inverse of the restriction functor. □

All of the above (definitions and) results carry over to the settings of (pre-)sheaves of (abelian) groups, rings, modules over a fixed ring, etc.

### (3.3) Categories and functors.

References: [GW1] Appendix A; [Alg2] Section 3.1[7].

A *category* $\mathcal{C}$ is given by a collection ("class") of objects $\mathrm{Ob}(\mathcal{C})$, for any two $X, Y \in \mathrm{Ob}(\mathcal{C})$ a collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms, for any object $X$ a morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, and for any three objects $X, Y, Z$ a map

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z), \quad (f,g) \mapsto f \circ g,$$

such that $f \circ \mathrm{id} = f$, $g \circ \mathrm{id} = g$, $(f \circ g) \circ h = f \circ (g \circ h)$ whenever these expressions are defined. We write $f\colon X \to Y$ if $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, and accordingly sometimes speak (and think) of morphisms in a category as

[7] https://math.ug/a2-ss23/subsec-kategorien.html

*arrows*. We sometimes write $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$. Set-theoretic remark: We usually implicitly make the assumption that for all $X$, $Y$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set (i.e., that $\mathcal{C}$ is what is usually called a *locally small* category).

*Examples.* The categories of sets (with maps of sets as morphisms), of finite sets, of groups (with group homomorphisms), of abelian groups, of rings (with ting homomorphisms), of modules over a fixed ring (with module homomorphisms), of finitely generated modules over a fixed ring, of topological spaces (with continuous maps as morphisms).

**Definition 3.11.** *Let $\mathcal{C}$ be a category. A morphism $f\colon X \to Y$ in $\mathcal{C}$ is called an* isomorphism, *if there exists a morphism $g\colon Y \to X$ in $\mathcal{C}$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.*

For objects $X, Y$ in $\mathcal{C}$ we say that $X$ and $Y$ are isomorphic and write $X \cong Y$, if there exists an isomorphism $X \to Y$ in $\mathcal{C}$.

Let $\mathcal{C}$, $\mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ is given by the following data: For each object $X$ of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$, and for every morphism $f\colon X \to Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(X) \to F(Y)$, such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X$ and such that $F(f \circ g) = F(f) \circ F(g)$.

It is useful to extend this notion in the following way. A *contravariant* functor $F$ from $\mathcal{C}$ to $\mathcal{D}$ is given by the following data: For each object $X$ of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$, and for every morphism $f\colon X \to Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(Y) \to F(X)$, such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X$ and such that $F(f \circ g) = F(g) \circ F(f)$.

In order to distinguish between the two sorts of functors, the first variant is called a *covariant* functor. A slightly different way to define (and denote) contravariant functor is as follows. Given a category $\mathcal{C}$, we define the *opposite* (or *dual*) category $\mathcal{C}^{\mathrm{opp}}$ as follows. It has the same objects as $\mathcal{C}$, and for any two objects $X, Y$, we set

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X),$$

i.e. “all arrows switch direction”. As identity morphisms we use the identity morphisms in $\mathcal{C}$. Composition in $\mathcal{C}^{\mathrm{opp}}$ is defined using the composition in $\mathcal{C}$ in the obvious way. Then a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a (covariant) functor $\mathcal{C}^{\mathrm{opp}} \to \mathcal{D}$. In view of this definition, one usually denotes contravariant functors in this way, i.e., as $\mathcal{C}^{\mathrm{opp}} \to \mathcal{D}$.

If $F$ is a functor and $f$ is an isomorphism, then $F(f)$ is an isomorphism.

The following properties of functors are often interesting, and we will need them later on.

**Definition 3.12.** *A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called*

(1) faithful, *if for all objects $X, Y \in \mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective,*
(2) full, *if for all objects $X, Y \in \mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective,*

(3) fully faithful, *if it is full and faithful, i.e., if for all objects* $X, Y \in \mathcal{C}$ *the map* $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ *is bijective,*

(4) essentially surjective, *if for every object* $Z \in \mathcal{D}$*, there exists an object* $X \in \mathcal{C}$ *such that* $F(X) \cong Z$ *(NB: isomorphism, not necessarily equality!).*

*For contravariant functors, there are analogous definitions.*

Next we define morphisms of functors (also called *natural transformations*).

**Definition 3.13.** *Let* $F, G\colon \mathcal{C} \to \mathcal{D}$ *be functors. A* morphism $\Phi\colon F \to G$ of functors *is given by a collection* $\Phi_X\colon F(X) \to G(X)$ *of morphisms in* $\mathcal{D}$ *for every* $X \in \mathcal{C}$*, such that for every morphism* $f\colon X \to Y$ *in* $\mathcal{C}$*, the diagram*

$$\begin{array}{ccc} F(X) & \xrightarrow{\Phi_X} & G(X) \\ \big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\Phi_Y} & G(Y) \end{array}$$

*commutes. (Applying the definition to functors* $\mathcal{C}^{\mathrm{opp}} \to \mathcal{D}$*, one similarly obtains the notion of morphism between two contravariant functors.)*

With this notion of morphism, together with the obvious identity morphisms and composition of morphisms of functors, the collection of all functors between fixed categories $\mathcal{C}$, $\mathcal{D}$ is itself a category, the *functor category* (however, the collection of all morphisms between two functors might not be a set). In particular, we also obtain the notion of *isomorphism between two*
Nov. 19, 2025 *functors* $\mathcal{C} \to \mathcal{D}$.

Functors are the natural "morphisms" between categories. In fact, we can define the category of all categories, where functors are the morphisms (again the collections of morphisms in this category are not necessarily sets). Note that we have obvious identity functors and can form the composition of functors. (Since we also defined morphisms between functors, there is, so to say, another level to the story in this case; this is formalized by the notion of *2-category*, but we will not have to go into this.) In particular, we obtain the notion of *isomorphism between categories*. However, it turns out that isomorphisms of categories are rather rare. A much more useful notion is the following weaker one.

**Proposition/Definition 3.14.** *A functor* $F\colon \mathcal{C} \to \mathcal{D}$ *is called an* equivalence of categories *if the following equivalent properties are satisfied.*

(i) *The functor* $F$ *has a* quasi-inverse $G$ *(i.e.,* $G$ *is a functor* $\mathcal{D} \to \mathcal{C}$ *such that* $G \circ F \cong \mathrm{id}_{\mathcal{C}}$*,* $F \circ G \cong \mathrm{id}_{\mathcal{D}}$*; it is crucial here that we only ask for isomorphisms, not equality, of these functors!).*

(ii) *The functor* $F$ *is fully faithful and essentially surjective.*

**Example 3.15.** Some (sketchy) examples of functors and morphisms of functors.

(1) $\mathrm{Spec}\colon (\mathrm{Rings})^{\mathrm{opp}} \to (\mathrm{Top})$

(2) forgetful functors

(3) Hom functors
(4) "adjointness tensor-Hom" as example of isomorphism of functors: for every $X$, have $\mathrm{Hom}(Y \otimes X, Z) \cong \mathrm{Hom}(Y, \mathrm{Hom}(X, Z))$ functorially in $Y$ and $Z$.
(5) dual vector space, morphism to double dual
(6) localization of a ring, base change (of modules or rings)
(7) $GL_n(-)$, $\det\colon GL_n(-) \to GL_1(-)$.

**Example 3.16.** Let $X$ be a topological space, and define the category $\mathrm{Ouv}(X)$ as follows. The objects of $\mathrm{Ouv}(X)$ are the open subsets of $X$. For open subsets $U, V \subseteq X$, we set

$$\mathrm{Hom}_{\mathrm{Ouv}(X)}(U, V) = \begin{cases} \{*\} & \text{if } U \subseteq V, \\ \emptyset & \text{otherwise.} \end{cases}$$

Here $\{*\}$ denotes a set with one element. There is then a unique way to define identity morphisms and composition, and one obtains a category.

With this definition, a presheaf of sets on $X$ is the same as a functor $\mathrm{Ouv}(X)^{\mathrm{opp}} \to (\mathrm{Sets})$. A morphism of presheaves is the same as a morphism of the corresponding functors. With this interpretation we in particular obtain a natural notion of presheaf on $X$ with values in any category $\mathcal{C}$ (namely a functor $\mathrm{Ouv}(X)^{\mathrm{opp}} \to \mathcal{C}$) and of morphisms between such presheaves (namely a morphism of the functors).

## (3.4) The structure sheaf of the spectrum of a ring.

We can now define the *structure sheaf on the spectrum of a ring*. So fix a ring $R$ and let $X = \mathrm{Spec}(R)$. We want to define a ("natural") sheaf of rings on $X$. As we have seen above, it is enough to define a sheaf (of rings) on the basis of the topology given by the principal opens, and we want to set $\mathscr{O}_X(D(f)) = R_f$.

The first step now is to check that this is well-defined (note that we may have $D(f) = D(g)$ for $f \neq g$).

**Lemma 3.17.**
(1) *For $f, g \in R$, we have $D(f) \subseteq D(g)$ if and only if $\frac{g}{1} \in R_f$ is a unit, i.e., $\frac{g}{1} \in R_f^\times$. In this case, we obtain a commutative diagram*

$$\begin{array}{ccc} R_g & \longrightarrow & R_f \\ & \nwarrow \quad \nearrow & \\ & R & \end{array}$$

*of ring homomorphisms (where $R \to R_f$ and $R \to R_g$ are the natural maps into the localizations).*
(2) *If $f, g \in R$ satisfy $D(f) = D(g)$, then there is a unique isomorphism $R_f \cong R_g$ of $R$-algebras.*

*Proof.* (1) We have

$$D(f) \subseteq D(g) \Leftrightarrow V(g) \subseteq V(f) \Leftrightarrow \sqrt{(f)} \subseteq \sqrt{(g)} \Leftrightarrow f \in \sqrt{(g)},$$

and this condition is equivalent to $\frac{g}{1} \in R_f^\times$. In fact, if $f^n = gh$, then $\frac{g}{1} \cdot \frac{h}{f^n} = 1$ in $R_f$. Conversely, if $\frac{g}{1} \cdot \frac{h'}{f^{n'}} = 1$ in $R_f$, then there is $m$ such that $gh'f^m = f^{n'+m}$.

The existence of the $R$-algebra homomorphism $R_g \to R_f$ then follows from properties of the localization of a ring (and in fact is also equivalent to the condition that $g$ maps to a unit in $R_f$). Furthermore, this homomorphism is the unique $R$-algebra homomorphism $R_f \to R_g$.

(2) follows from (1). □

Using Part (2) of the lemma, we see that we can attach to each principal open $D(f)$ the ring $R_f$ (well-defined up to *unique* isomorphism, so we can identify the rings $R_f$, $R_g$ for $D(f) = D(g)$ in a specific way). Alternatively, the lemma implies that we have a unique isomorphism

$$R_f \cong S^{-1}R \quad \text{for } S = \{g \in R;\ D(f) \subseteq D(g)\},$$

of $R$-algebras, and the right hand side $S^{-1}R$ only depends on $D(f)$, not on $f$.

**Definition 3.18.** *Let $R$ be a ring, $X = \operatorname{Spec}(R)$, $\mathcal{B}$ the basis of the Zariski topology on $X$ given by all principal open subsets. We define a presheaf $\mathscr{O}'_X$ on $\mathcal{B}$ by setting*

$$\mathscr{O}'_X(D(f)) = R_f$$

*and as restriction maps $\mathscr{O}'_X(D(g)) \to \mathscr{O}'_X(D(f))$ for $D(f) \subseteq D(g)$ use the unique $R$-algebra homomorphism $R_g \to R_f$ (cf. Lemma 3.17).*

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**Lemma 3.19.** *Let $R$ be a ring, $f_i \in R$, $i \in I$. Then $\bigcup_{i\in I} D(f_i) = \operatorname{Spec}(R)$ if and only the elements $f_i$ generate the unit ideal.*

*Proof.* The condition $\bigcup_{i\in I} D(f_i) = \operatorname{Spec}(R)$ is equivalent to saying that the ideal $(f_i;\ i \in I)$ is not contained in any prime ideal, but then the quotient $R/(f_i;\ i \in I)$ cannot have a maximal ideal, so is the zero ring. □

Note that the lemma proves that $\operatorname{Spec}(R)$ is quasi-compact. It also shows that whenever $f_1, \dots, f_r \in R$ generate the unit ideal, then for every $N \geq 1$, also $f_1^N, \dots, f_r^N$ generate the unit ideal.

**Theorem 3.20.** *The presheaf $\mathscr{O}'_X$ on $\mathcal{B}$ of Definition 3.18 is a sheaf.*

We denote the sheaf on $X$ that we obtain from $\mathscr{O}'_X$ by Proposition 3.10 by $\mathscr{O}_X$ and call it the *structure sheaf on $X$*.

*Proof.* We need to show: For all $f, f_i \in R$ such that $D(f) = \bigcup_{i\in I} D(f_i)$, the sequence

$$0 \to R_f \xrightarrow{\rho} \prod_i R_{f_i} \xrightarrow{\sigma} \prod_{i,j} R_{f_i f_j},$$

where the maps are $\rho(s) = \left(\frac{s}{1}\right)_i$ (with the $i$-th entry in $R_{f_i}$), and $\sigma((s_i)_i) = \left(\frac{s_i}{1} - \frac{s_j}{1}\right)_{i,j}$, is exact.

We first do the following reduction steps:

- Replacing $R$ by $R_f$, we may assume that $f = 1$, and hence that $R_f = R$.
- Since all principal open subsets are quasi-compact, we may assume that the index set $I$ of the open cover if *finite*. (This requires a small "computation".)

As the previous lemma shows, the assumption that $\bigcup_i D(f_i) = \operatorname{Spec}(R)$ is equivalent to saying that the elements $f_i$ generate the unit ideal in $R$. This implies that for every $N$, the powers $f_i^N$ also generate the unit ideal. We will refer to this property by (*).

*Injectivity in the above sequence.* Let $s \in R$ such that the image of $s$ in each localization $R_{f_i}$ vanishes. Then for each $i$ there exists $N_i$ such that $f^{N_i}s = 0$. Since $I$ is finite, we find $N$ with the property that $f_i^N s = 0$ for all $i$. Now use (*) to write $1 = \sum_i g_i f_i^N$. We then see that

$$s = \left(\sum_i g_i f_i^N\right) s = 0.$$

*Exactness "in the middle":* $\operatorname{Im}(\rho) = \operatorname{Ker}(\sigma)$. The inclusion $\subseteq$ is clear (in fact, it holds for any presheaf). So let $(s_i)_i \in \operatorname{Ker}(\sigma)$. We write

$$s_i = \frac{a_i}{f_i^N}$$

(again we use that $I$ is finite, so that we can find an $N$ that works for all $s_i$).

The assumption that $(s_i)_i \in \operatorname{Ker}(\sigma)$ means that all the differences $\frac{s_i}{1} - \frac{s_j}{1} \in R_{f_i f_j}$ vanish, so we find $M \geq 0$ such that

$$(f_i f_j)^M (f_j^N a_i - f_i^N a_j) = 0.$$

Now we use (*) to write $1 = \sum g_i f_i^{M+N}$ (these are other $g_i$'s than above).

Define $a = \sum_j g_j f_j^M a_j$. We will check that $\rho(a) = (s_i)_i$. For this we need to prove that $\frac{a}{1} - \frac{a_i}{f_i^N} = 0 \in R_{f_i}$ for all $i$. But from the definition of $a$ it follows that

$$a f_i^{M+N} = \sum_j g_j f_j^M a_j f_i^{M+N} = \sum_j g_j f_i^M a_i f_j^{M+N} = a_i f_i^M,$$

and that implies the result. □

**(3.5) Stalks.**

References: For foundational material on the notion of colimit, see [GW1] Appendix A (and the problem sheets).

Let $\mathscr{F}$ be a presheaf on a topological space $X$, and let $x$. The "stalk" of the sheaf is the collection of all sections defined on some (possibly very small) open neighborhood of $x$, in the following precise sense.

**Definition 3.21.** *Let $X$ be a topological space and let $x \in X$. For every presheaf $\mathscr{F}$ on $X$ we define the* stalk of $\mathscr{F}$ at $x$ *as*

$$\mathscr{F}_x = \operatorname*{colim}_{x\in U} \mathscr{F}(U) = \left( \coprod_{x\in U} \mathscr{F}(U) \right) / \sim,$$

*where the colimit is taken over the restriction maps of $\mathscr{F}$ as transition maps (i.e., the partially ordered index set is the set of all open subsets of $X$ containing $x$, ordered by $V \le U :\Leftrightarrow U \subseteq V$). We can construct the colimit explicitly as written on the right hand side. The equivalence relation $\sim$ is defined by setting, for $s \in \mathscr{F}(U)$, $t \in \mathscr{F}(V)$,*

$$s \sim t \quad \Longleftrightarrow \quad \text{there exists an open } W \subseteq U \cap V, x \in W, \text{ such that } s_{|W} = t_{|W}.$$

For presheaves with values in some category $\mathcal{C}$ (rather than sets), we would take the colimit above in the category $\mathcal{C}$ (assuming that it exists). However, the following argument shows that for the categories that we will be concerned with in this class, this distinction is not important.

The index set of the colimit is filtered (for $U$, $V$ open neighborhoods of $x$, we have $U \cap V \subseteq U, V$, so $U, V \le U \cap V$, and $U \cap V$ is again an open neighborhood of $x$). Therefore, if $\mathscr{F}$ is a presheaf of groups / abelian groups / rings, then the stalk of $\mathscr{F}$ at a point $x$ in the category of sets (i.e., as constructed above) has a natural structure of group / abelian group / ring, etc., and this gives the colimit in the category of groups / etc. For every open neighborhood $U$ of $x$, we have a natural map $\mathscr{F}(U) \to \mathscr{F}_x$; if $\mathscr{F}$ is a presheaf of groups (etc.), then this is a group homomorphism (etc.).

If $\mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, for all open neighborhoods $U \subseteq V$ of $x$ we have a commutative diagram

$$\begin{array}{ccc}
\mathscr{F}(V) & \longrightarrow & \mathscr{G}(V) \\
\downarrow & & \downarrow \\
\mathscr{F}(U) & \longrightarrow & \mathscr{G}(U) \\
\downarrow & & \downarrow \\
\mathscr{F}_x & & \mathscr{G}_x.
\end{array}$$

Here the vertical maps are the restriction map and the natural maps to the stalk.

These diagrams induce a morphism $\mathscr{F}_x \to \mathscr{G}_x$ between the stalks. This shows that the construction of stalks is a functor from the category of presheaves (of sets) to the category of sets. Likewise, we obtain functors from the category of presheaves of abelian groups to the category of abelian groups, and similarly for presheaves of groups, rings, etc.

**Example 3.22.** In the theory of holomorphic functions (complex analysis in one or several variables), the stalk of the structure sheaf can be interpreted in terms of convergent power series as follows. Let $X = \mathbb{C}$ (or more generally an open subset of $\mathbb{C}^n$, $n \geq 1$, or even more generally any complex manifold), and define the "structure sheaf" $\mathscr{O}_X$ by setting

$$\mathscr{O}_X(U) = \{f : U \to \mathbb{C} \text{ holomorphic}\}.$$

Then the stalk $\mathscr{O}_{X,x}$ can be identified with the ring of convergent power series at $x$ (i.e., power series in $n$ variables that converge in some open neighborhood of $x$). The *identity theorem* can be phrased as saying that for every connected open neighborhood $U$ of $x$, the natural map $\mathscr{O}_X(U) \to \mathscr{O}_{X,x}$ is injective.

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**Remark 3.23.** Recall that for the computation of the limit of a convergent sequence of real numbers, we may pass to a subsequence, i.e., we may replace the index set $\mathbb{N}$ of the limit by any infinite subset of $\mathbb{N}$. Something similar holds for colimits:

Let $(I, \leq)$ be a partially ordered set. We call a subset $I' \subseteq I$ *cofinal*, if for every $i \in I$ there exists $i' \in I'$ with $i \leq i'$. We equip $I'$ with the induced partial order. Then for every inductive system $(F_i)_{i\in I}$, we have a natural isomorphism

$$\operatorname*{colim}_{i\in I'} F_i \cong \operatorname*{colim}_{i\in I} F_i.$$

**Example 3.24.** Let $R$ be a ring, $X = \operatorname{Spec}(R)$. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Let us compute the stalk of the structure sheaf $\mathscr{O}_X$ at the point $\mathfrak{p}$. By the previous remark, we may compute the stalk as

$$\mathscr{O}_{X,\mathfrak{p}} = \operatorname*{colim}_{f\in R,\ \mathfrak{p}\in D(f)} \mathscr{O}_X(D(f)) = \operatorname*{colim}_{f\in R,\ \mathfrak{p}\in D(f)} R_f.$$

The latter colimit is isomorphic to the localization $R_\mathfrak{p}$ of $R$ with respect to $\mathfrak{p}$ (i.e., the localization with respect to the open subset $R\backslash\mathfrak{p}$). In fact, the universal property of the colimit gives us a ring homomorphism $\operatorname*{colim}_{f\in R,\ \mathfrak{p}\in D(f)} R_f \to R_\mathfrak{p}$ (here we use that $\mathfrak{p} \in D(f)$ by definition is equivalent to $f \notin \mathfrak{p}$, and in this case $\frac{f}{1}$ is a unit in $R_\mathfrak{p}$).

It is easy to see that this map is surjective. For the injectivity, we need to show that for all $s, f \in R$, $\mathfrak{p} \in D(f)$ and $i \geq 0$, $\frac{s}{f^i} = 0$ in $R_\mathfrak{p}$ implies that $\frac{s}{f^i} = 0$ in some localization $R_{fg}$ with $\mathfrak{p} \in D(g)$. This also follows immediately from properties of the localization.

The following propositions illustrate in which sense the stalks capture "local information about a sheaf".

**Lemma 3.25.** *Let $X$ be a topological space, $\mathscr{F}$ a sheaf on $X$, $U \subseteq X$ an open subset. Then the natural map*

$$\mathscr{F}(U) \to \prod_{x\in U} \mathscr{F}_x$$

*is injective.*

*Proof.* Let $s \in \mathscr{F}(U)$ be an element which maps to 0 in every stalk $\mathscr{F}_x$, $x \in U$. By definition of the stalk, thus for every $x$, there exists an open neighborhood $V_x \subseteq U$ of $x$ such that $s_{V_x} = 0$ (in $\mathscr{F}(V_x)$). But then clearly all the $V_x$ cover $U$, and the sheaf axioms imply that $s = 0$. □

**Proposition 3.26.** *Let $X$ be a topological space, and let $\mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on $X$.*

(1) *The following are equivalent:*
  (i) *For every open $U \subseteq X$, the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is injective.*
  (ii) *For every $x \in X$, the map $\mathscr{F}_x \to \mathscr{G}_x$ is injective.*

(2) *The following are equivalent:*
  (i) *For every open $U \subseteq X$, the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is bijective.*
  (ii) *The morphism $\mathscr{F} \to \mathscr{G}$ is an isomorphism.*
  (iii) *For every $x \in X$, the map $\mathscr{F}_x \to \mathscr{G}_x$ is bijective.*

*Proof.* (1) The implication (ii) ⇒ (i) follows from the previous lemma. For the implication (i) ⇒ (ii), let $x \in X$, and $s \in \mathscr{F}_x$ mapping to 0 in $\mathscr{G}_x$. Let $\dot{s} \in \mathscr{F}(V)$ be a representative of $s$, i.e., an element with image $s$, where $V$ is a suitable open neighborhood of $x$. The image of $\dot{s}$ in $\mathscr{G}(V)$ maps to 0 in the stalk at $x$, hence its restriction to a suitable open neighborhood $U$ of $x$ is 0 (in $\mathscr{G}(U)$). But then the injectivity in (i) implies that $\dot{s}_{|U} = 0$, and a fortiori $s = 0$.

(2) See Problem sheet 6. □

Note that the analogous statement to (1) for *surjective* maps *is not true*! (Cf. Problem sheet 7.) It turns out that the stalks provide the correct perspective on these properties, and we make the following definition.

**Definition 3.27.** *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a topological space $X$. We call $\varphi$* injective *(*surjective*,* bijective*, respectively), if for every $x \in X$ the map $\mathscr{F}_x \to \mathscr{G}_x$ induced by $\varphi$ is* injective *(*surjective*,* bijective*, respectively).*

**Proposition 3.28.** *Let $X$ be a topological space and let $\varphi, \psi\colon \mathscr{F} \to \mathscr{G}$ be two morphisms between the sheaves $\mathscr{F}$ and $\mathscr{G}$ on $X$. The following are equivalent:*

(i) *The morphisms $\varphi$, $\psi$ are equal.*
(ii) *For every $x \in X$, the induced morphisms $\varphi_x, \psi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ between the stalks are equal.*

*Proof.* It is clear that (i) implies (ii). The converse follows from Lemma 3.25 □

**(3.6) Sheafification.**

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We now study a "natural way" of attaching to an arbitrary presheaf a *sheaf* that is "as close as possible" to the given presheaf (in particular, both have the same stalk at each point of the underlying space). This sheaf will be called the *sheafification* (German: *Garbifizierung*) of the given presheaf. We will define it by a universal property, as follows:

**Definition/Proposition 3.29.** *Let $X$ be a topological space and let $\mathscr{F}$ be a presheaf on $X$.*

(1) *A sheaf $\widetilde{\mathscr{F}}$ together with a morphism $\iota_{\mathscr{F}}\colon \mathscr{F} \to \widetilde{\mathscr{F}}$ of presheaves is called a* sheafification *of $\mathscr{F}$, if for every morphism $\mathscr{F} \to \mathscr{G}$ from $\mathscr{F}$ to a sheaf $\mathscr{G}$, there exists a unique morphism $\widetilde{\mathscr{F}} \to \mathscr{G}$ such that the diagram*

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathscr{G} \\ & \searrow \quad \nearrow & \\ & \widetilde{\mathscr{F}} & \end{array}$$

*commutes.*

(2) *A sheafification of the presheaf $\mathscr{F}$ exists. If $\mathscr{F}$ is a presheaf of groups, abelian groups, rings, ..., then so is the sheafification.*

(3) *The morphism $\mathscr{F} \to \widetilde{\mathscr{F}}$ induces an isomorphism on the stalks for each $x \in X$.*

(4) *For every morphism $\mathscr{F} \to \mathscr{G}$ we obtain a morphism $\widetilde{\mathscr{F}} \to \widetilde{\mathscr{G}}$, so that the diagram*

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathscr{G} \\ \downarrow & & \downarrow \\ \widetilde{\mathscr{F}} & \longrightarrow & \widetilde{\mathscr{G}} \end{array}$$

*commutes. Thus sheafification defines a functor from the category of presheaves on $X$ to the category of sheaves on $X$.*

It follows immediately from the definition that the sheafification is uniquely determined up to isomorphism, and that the sheafification of a sheaf $\mathscr{F}$ is simply the identity morphism.

*Proof.* We need to prove the existence of the sheafification. We may construct it explicitly as follows. For $U \subseteq X$ open, we set

$$\widetilde{\mathscr{F}}(U) = \Big\{ (s_x)_x \in \prod_{x \in U} \mathscr{F}_x;$$

$$\text{for all } x \text{ there ex. } x \in W \subseteq U \text{ open}, t \in \mathscr{F}(W), \text{ s.t. for all } w \in W\colon s_w = t_w \Big\}.$$

(This is a natural candidate in view of Lemma 3.25.) The restriction maps are defined as the obvious projection maps. One checks that this defines a sheaf. From this points, the rest of the proof is not difficult, but doing things in the right order saves some work. We have a morphism $\mathscr{F} \to \widetilde{\mathscr{F}}$ of presheaves by defining

$$\mathscr{F}(U) \to \widetilde{\mathscr{F}}(U), \quad s \mapsto (s_x)_x,$$

where $s_x$ denotes the image of $s$ in the stalk $\mathscr{F}_x$. At this point one checks that this induces an isomorphism on all stalks (thus proving Part (3)).

For $\varphi\colon \mathscr{F} \to \mathscr{G}$ a morphism of presheaves we obtain a morphism $\widetilde{\mathscr{F}} \to \widetilde{\mathscr{G}}$ by setting

$$\widetilde{\mathscr{F}}(U) \to \widetilde{\mathscr{G}}(U), \quad (s_x)_x \mapsto (\varphi_x(s_x))_x.$$

By Part (3) and Proposition 3.28, this is the unique morphism which makes the diagram in (4) commutative. Applying this in the special case where the target $\mathscr{G}$ of the morphism $\mathscr{F} \to \mathscr{G}$ is a sheaf, shows the universal property, because in that case $\mathscr{G} = \widetilde{\mathscr{G}}$ (more precisely, the morphism $\mathscr{G} \to \widetilde{\mathscr{G}}$ is an isomorphism, by Proposition 3.26. □

**Example 3.30.** (constant sheaf) Let $X$ be a topological space, $E$ a set (or abelian group, …). We define the presheaf $\mathscr{F}$ by setting

$$\mathscr{F}(U) = E, \quad U \subseteq X \text{ open},$$

with the identity maps as restriction maps. The *constant sheaf with value $E$* is defined as the sheafification of this presheaf. All its stalks are naturally equal to $E$. (But in general, the sections on an open $U$ of $X$ may be different from $E$.)

A typical use of the sheafification is to do certain constructions of sheaves that can be naturally carried out for presheaves, but may not themselves yield sheaves (even if one starts with sheaves, so to say), as in the following example.

**Example 3.31.** Let $f\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. The *image sheaf* $\mathrm{Im}(f)$ is defined as the sheafification of the presheaf

$$U \mapsto \mathrm{Im}(\mathscr{F}(U) \to \mathscr{G}(U)).$$

(Note that this presheaf usually is not a sheaf!)

The image sheaf comes with a natural injective sheaf morphism $\mathrm{Im}(f) \to \mathscr{G}$. This is an isomorphism if and only of $f$ is surjective.

**Remark 3.32.** We can express the universal property of the sheafification by saying that, for every presheaf $\mathscr{F}$ and sheaf $\mathscr{G}$, there are bijections

$$\mathrm{Hom}(\widetilde{\mathscr{F}}, \mathscr{G}) \xrightarrow{\cong} \mathrm{Hom}(\mathscr{F}, \mathscr{G})$$

(given by composition with the natural map $\mathscr{F} \to \widetilde{\mathscr{F}}$; these maps are functorial in $\mathscr{F}$ and in $\mathscr{G}$). On the right, we "view $\mathscr{G}$ as a presheaf" (forgetting that it is a sheaf), and have Hom in the category of presheaves. On the left, we have Hom in the category of sheaves, because source and target are sheaves. (Since morphisms of sheaves by definition are just morphisms of presheaves, this may sound overly pedantic, but at this point it is useful to distinguish between the two categories.)

We can thus express this by saying that the sheafification functor is left adjoint to the inclusion functor from the category of sheaves into the category of presheaves.

### (3.7) Direct and inverse image.

Let $f\colon X \to Y$ be a continuous map between topological spaces. We want to think about how we could "transport" sheaves from $X$ to $Y$ and vice versa.

Given a presheaf $\mathscr{F}$ on $X$, it is fairly straighforward to define its "pushforward" or "direct image" in $Y$.

**Definition 3.33.** *Let $f\colon X \to Y$ be a continuous map and let $\mathscr{F}$ be a presheaf on $X$. We define its* direct image *(or* pushforward*) $f_*\mathscr{F}$ by*

$$(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V)), \quad V \subseteq Y \text{ open}$$

*with the restriction maps induced by the restriction maps of $\mathscr{F}$.*

It is easy to check that for a sheaf $\mathscr{F}$, the direct image $f_*\mathscr{F}$ is again a sheaf. While this construction is fairly simple, note that it is usually not easy to express the stalks of $f_*\mathscr{F}$ in terms of the stalks of $\mathscr{F}$.

If $\mathscr{F} \to \mathscr{F}'$ is a morphism of sheaves on $X$, then we obtain a morphism $f_*\mathscr{F} \to f_*\mathscr{F}'$ in the obvious way. It is easy to check that $f_*$ defines a functor from the category of sheaves on $X$ to the category of sheaves on $Y$. Similarly for sheaves of groups, abelian groups, rings, etc.

If $f\colon X \to Y$, $g\colon Y \to Z$ are continuous maps, and $\mathscr{F}$ is a presheaf on $X$, then $(g \circ f)_*\mathscr{F} = g_*f_*\mathscr{F}$.

Going in the other direction is a little more cumbersome, but with the sheafification we have all the necessary tools at our disposal. We will proceed in two steps and first define a presheaf.

**Definition 3.34.** *Let $f\colon X \to Y$ be a continuous map and let $\mathscr{G}$ be a presheaf on $Y$. We define its* inverse image presheaf *$f^+\mathscr{G}$ by*

$$(f^+\mathscr{G})(U) = \operatorname*{colim}_{f(U)\subseteq V} \mathscr{G}(V),$$

*where the colimit is taken along the restriction maps of $\mathscr{G}$, and the restriction maps are obtained from the restriction maps of $\mathscr{G}$ and the universal property of the colimit.*

Even if $\mathscr{G}$ is a sheaf, the presheaf $f^+\mathscr{G}$ usually is not a sheaf, and we define

**Definition 3.35.** *Let $f\colon X \to Y$ be a continuous map and let $\mathscr{G}$ be a presheaf on $Y$. We define its* inverse image sheaf *(or* pullback*) $f^{-1}\mathscr{G}$ to be the sheafification of $f^+\mathscr{G}$.*

The construction $\mathscr{G} \mapsto f^+\mathscr{G}$ is functorial in $\mathscr{G}$. Since sheafification is also functorial in $\mathscr{G}$, we see that the inverse image $f^{-1}$ is a functor from the category of presheaves on $Y$ to the category of sheaves on $X$. If $\mathscr{G}$ is a presheaf of (abelian) groups, rings, etc., then so are $f^+\mathscr{G}$ and $f^{-1}\mathscr{G}$.

**Example 3.36.** Let $X$ be a topological space, $\mathscr{F}$ a presheaf on $X$, $x \in X$, and $i\colon \{x\} \to X$ the inclusion map. Then $i^{-1}\mathscr{F}$ is the sheaf on the one-point set $\{x\}$ with sections (on the whole space) $\mathscr{F}_x$.

It is not difficult to compute the stalks of the inverse image presheaf:

**Proposition 3.37.** *Let $f\colon X \to Y$ be a continuous map and let $\mathscr{G}$ be a presheaf on $Y$. Let $x \in X$. There is a natural isomorphism*

$$(f^{-1}\mathscr{G})_x \cong \mathscr{G}_{f(x)}.$$

*These isomorphisms are functorial in $\mathscr{G}$.*

*Proof.* We have

$$(f^{-1}\mathscr{G})_x = (f^+\mathscr{G})_x = \operatorname*{colim}_{x\in U\subseteq X} \operatorname*{colim}_{f(U)\subseteq V\subseteq Y} \mathscr{G}(V) = \mathscr{G}_{f(x)}.$$

For the final equality we use that $f$ is continuous and that therefore every open neighborhood $V \subseteq Y$ of $f(x)$ contains $f(U)$ for some $U$ (e.g., $U = f^{-1}(V)$). □

From the computation of the stalks, we obtain the compatibility with composition of morphisms in the following sense.

**Proposition 3.38.** *Let $f\colon X \to Y$, $g\colon Y \to Z$ be continuous maps and let $\mathscr{H}$ be a presheaf on $Z$. Then there is a natural isomorphism*

$$(g \circ f)^{-1}\mathscr{H} \cong f^{-1}(g^{-1}\mathscr{H}).$$

*These isomorphisms are functorial in $\mathscr{H}$.*

*Proof.* One first checks that $(g\circ f)^+\mathscr{H} \cong f^+(g^+\mathscr{H})$. Passing to the sheafification, this implies that $(g\circ f)^{-1}\mathscr{H} \cong f^{-1}(g^+\mathscr{H})$. On the other hand, (the proof of) Proposition 3.37 implies that $f^+(g^+\mathscr{H}) \cong f^{-1}(g^+\mathscr{H})$, because the natural morphism between these sheaves induces isomorphisms on all stalks. □

Finally we note the following useful relation between the functors $f_*$ and $f^{-1}$.

**Proposition 3.39.** *Let $f\colon X \to Y$ be a continuous map, and let $\mathscr{F}$ be a sheaf on $X$, $\mathscr{G}$ a presheaf on $Y$. Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{PreSh}(Y)}(\mathscr{G}, f_*\mathscr{F}).$$

*These isomorphisms are functorial in $\mathscr{F}$ and in $\mathscr{G}$. In other words, $f^{-1} \dashv f_*$ is a pair of adjoint functors.*

In the situation of the proposition, of $\mathscr{F}$ is a sheaf of abelian groups and $\mathscr{G}$ is a presheaf of abelian groups, then the Hom sets above are abelian groups and the adjunction isomorphisms are group isomorphisms.

*Proof.* First note that since $\mathscr{F}$ is a sheaf, we may identify $\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F})$ with $\mathrm{Hom}_{\mathrm{PreSh}(X)}(f^{+}\mathscr{G}, \mathscr{F})$. It is then not difficult to explicitly construct natural maps in both directions and to check that they are inverse to each other and functorial. □

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