Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2022/23 Dr. Heer Zhao

Problem sheet 7

Due date: Nov. 29, 2022.

Problem 25

Let X be a topological space and let $\mathscr{F}$, $\mathscr{G}$ be sheaves of abelian groups on X. For every open $U \subseteq X$, denote by $\mathrm{Hom}(\mathscr{F}_{|U}, \mathscr{G}_{|U})$ the abelian group of morphisms $\mathscr{F}_{|U} \to \mathscr{G}_{|U}$ of sheaves of abelian groups on U. This defines a presheaf in a natural way. Show that this presheaf is a sheaf.

Problem 26

Let X be a topological space and let $\mathscr{F}$ be a presheaf on X. For $x \in X$, we denote by $\mathscr{F}_x$ the stalk of $\mathscr{F}$ at x, and for an open neighborhood U of x and a section $s \in \mathscr{F}(U)$, we denote by $s_x \in \mathscr{F}_x$ the image of s under the natural map $\mathscr{F}(U) \to \mathscr{F}_x$. We define a presheaf $\widetilde{\mathscr{F}}$ on X by setting

$$\widetilde{\mathscr{F}}(U) = \left\{ (s_x)_x \in \prod_{x \in U} \mathscr{F}_x;\ \forall x \in U \exists x \in W \subseteq U \text{ open},\ t \in \mathscr{F}(W)\colon \forall w \in W\colon s_w = t_w \right\}$$

for $U \subseteq X$ open, and where the restriction maps $\widetilde{\mathscr{F}}(U) \to \widetilde{\mathscr{F}}(V)$ (for $V \subseteq U \subseteq X$ open) are given as the restriction of the projection $\prod_{x \in U} \mathscr{F}_x \to \prod_{x \in V} \mathscr{F}_x$ to $\widetilde{\mathscr{F}}(U)$.

(1) Prove that $\widetilde{\mathscr{F}}$ is a sheaf.

(2) Let $U \subseteq X$ be open. Show that the image of the map $\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x$, $s \mapsto (s_x)_x$, lies in $\widetilde{\mathscr{F}}(U)$. Prove that this defines a morphism $\iota_{\mathscr{F}}\colon \mathscr{F} \to \widetilde{\mathscr{F}}$ of presheaves.

(3) Let $x \in X$. Prove that the map $\mathscr{F}_x \to \widetilde{\mathscr{F}}_x$ on the stalks induced by $\iota_{\mathscr{F}}$ is bijective.

Problem 27

Let X be an irreducible topological space, E a set, and $\mathscr{F}$ the constant sheaf on X associated with E. Show that $\mathscr{F}(U) = E$ for every non-empty open set $U \subseteq X$.

Problem 28

Let X be a topological space and let $f\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X.

(1) We define the *image* $\operatorname{im}(f)$ *of* f as the sheafification of the presheaf

$$U \mapsto \operatorname{im}(\mathscr{F}(U) \to \mathscr{G}(U)), \quad U \subseteq X \text{ open}.$$

Prove that f induces a surjective morphism $\mathscr{F} \to \operatorname{im}(f)$ of sheaves.

(2) Now assume that f above is a morphism of sheaves of abelian groups. We define the *kernel* $\ker(f)$ *of* f as the sheaf

$$U \mapsto \ker(\mathscr{F}(U) \to \mathscr{G}(U)), \quad U \subseteq X \text{ open}.$$

Prove that this is in fact a sheaf and that f induces an injective morphism $\ker(f) \to \mathscr{F}$.