Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2022/23 Dr. Heer Zhao

Problem sheet 6

Due date: Nov. 22, 2022.

Problem 21

Let X be a topological space and let $\mathscr{F}$ be a sheaf on X. Let $U \subseteq X$ be an open subset and $s, t \in \mathscr{F}(U)$. Show that the set of $x \in U$ with $s_x = t_x \in \mathscr{F}_x$ is open in X.

Problem 22

Let X be a topological space and let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that the following are equivalent:

(i) for all open subsets $U \subseteq X$, the map $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is bijective,

(ii) the morphism φ is an isomorphism of sheaves,

(iii) for all $x \in X$, the map $\varphi\colon \mathscr{F}_x \to \mathscr{G}_x$ on the stalks is bijective.

Problem 23

Give an example of a topological space X, a surjective map $\mathscr{F} \to \mathscr{G}$ of sheaves on X and an open $U \subseteq X$ such that the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.

Problem 24

Let X be a topological space and let $(U_i)_i$ be an open covering of X. For all i let $\mathscr{F}_i$ be a sheaf on U_i. Assume that for each pair (i, j) of indices we are given isomorphisms $\varphi_{ij}\colon \mathscr{F}_{j|U_i \cap U_j} \xrightarrow{\sim} \mathscr{F}_{i|U_i \cap U_j}$ satisfying for all i, j, k the "cocycle condition"

$$\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk} \quad \text{on } U_i \cap U_j \cap U_k.$$

Show that there exists a sheaf $\mathscr{F}$ on X and for all i isomorphisms $\psi_i\colon \mathscr{F}_i \xrightarrow{\sim} \mathscr{F}_{|U_i}$ such that $\psi_i \circ \varphi_{ij} = \psi_j$ on $U_i \cap U_j$ for all i, j. Show that $\mathscr{F}$ and the ψ_i are uniquely determined up to unique isomorphism by these conditions.

Remark: We say that the sheaf $\mathscr{F}$ is obtained by *gluing the* $\mathscr{F}_i$ via the *gluing datum* φ_{ij}.