Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2022/23 Dr. Heer Zhao

Problem sheet 3

Due date: Nov. 01, 2022.

Problem 9 Let $\varphi\colon A \to B$ be a ring homomorphism, and let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ denote the map attached to φ.

(a) Let $\mathfrak{b} \subseteq B$ be an ideal. Prove that

$$\overline{f(V(\mathfrak{b}))} = V(\varphi^{-1}(\mathfrak{b}))$$

(where $\bar{\cdot}$ denotes closure).

(b) Assume that φ is surjective. Prove that f induces a homeomorphism from $\operatorname{Spec} B$ onto $V(\ker(\varphi))$.

(c) Prove that the image of f is dense in $\operatorname{Spec} A$ if and only if every element of $\ker(\varphi)$ is nilpotent.

Problem 10 Let A be a ring, $f \in A$ and $X := D(f) \subseteq \operatorname{Spec} A$ with the subspace topology. Prove that the topological space X is quasi-compact (i.e., for every cover $X = \bigcup_{i\in I} U_i$ by open subsets U_i, there exists a finite subset $I' \subseteq I$ with $X = \bigcup_{i\in I'} U_i$).

Problem 11 Let $(I, \leqslant)$ be a partially ordered set. An *inductive system* of sets (with index set I) is a family X_i, $i \in I$, of sets together with maps $\varphi_{ji}\colon X_i \to X_j$ for all pairs $i, j \in I$ with $i \leqslant j$, such that

$$\varphi_{ii} = \mathrm{id}_{X_i}, \quad \varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$$

for all $i \leqslant j \leqslant k \in I$.

A set C together with maps $\psi_i\colon X_i \to C$ such that $\psi_j \circ \varphi_{ji} = \psi_i$ for all $i \leqslant j$ is called a *colimit* (or *direct limit* or *inductive limit*) if it satisfies the following "universal property": For every set T together with maps $\xi_i\colon X_i \to T$ such that $\xi_j \circ \varphi_{ji} = \xi_i$ for all $i \leqslant j$, there exists a unique map $\chi\colon C \to T$ such that $\chi \circ \psi_i = \xi_i$ for all i. The colimit is also denoted by $\operatorname{colim}_{i\in I} X_i$ or by $\varinjlim_{i\in I} X_i$. (The maps φ_{ij} and ψ_i are usually omitted from the notation.)

(a) Suppose that I is *directed*, i.e., for all $i, j \in I$ there exists $k \in I$ with $i \leqslant k$ and $j \leqslant k$. Let (X_i, φ_{ji}) be an inductive system of sets, let U be the disjoint union

$$U = \coprod_{i\in I} X_i,$$

and consider the following relation on U: for $x, y \in U$, say $x \in X_i$, $y \in X_j$, we set $x \sim y$ if and only if there exists $k \geqslant i, j$ with $\varphi_{ki}(x) = \varphi_{kj}(y)$. Prove that $\sim$ is an equivalence relation and that the set $U/\sim$ of equivalence classes together with the natural maps $X_i \to U/\sim$ is a colimit of the system (X_i, φ_{ji}).

(b) Assume that I is directed, that all X_i are subsets of a set X, that $i \leqslant j$ if and only if $X_i \subseteq X_j$, and that the maps φ_{ji} are the inclusion maps. Prove that the union $C := \bigcup_i X_i$ with the inclusion maps $X_i \to C$ is a colimit of the X_i.

Problem 12 Let I be a partially ordered set.

(a) Define the notion of *colimit* (with index set I) in an arbitrary category.

(b) Suppose that I is directed. Prove that all colimits with index set I in the category of abelian groups exist.

(c) Suppose that I is directed. Prove that the functor colim_i on the category of abelian groups is exact, i.e.: Let (A_i, φ_{ji}), (B_i, ψ_{ji}), (C_i, ξ_{ji}) be inductive systems of abelian groups indexed by I. Suppose that for each i, there are short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0$$

which are compatible with the maps φ_{ji}, ψ_{ji}, ξ_{ji} in the obvious sense. Prove that these sequences induce a sequence

$$0 \to \operatorname*{colim}_i A_i \to \operatorname*{colim}_i B_i \to \operatorname*{colim}_i C_i \to 0$$

which is again exact.