Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2022/23 Dr. Heer Zhao

Problem sheet 2

Due date: Oct. 25, 2022.

Problem 5

Let k be an algebraically closed field. Let $n \geqslant 1$. We identify the space $M := \mathrm{Mat}_{n\times n}(k)$ of $(n \times n)$-matrices with entries in k with k^{n^2} and equip it with the Zariski topology. By Problem 2 (2), it is irreducible.

(1) Prove that the subset of M consisting of matrices A such that $\mathrm{charpol}_A(A) = 0$ is closed in M (without using the Theorem of Cayley-Hamilton).

(2) Use Problem 4 (including the remark that the result holds for general n) to prove that the subset of diagonalizable matrices with n different eigenvalues in k is open in M.

(3) Prove the Theorem of Cayley-Hamilton, i.e., prove that the subset defined in (1) equals all of M.

Problem 6 Let k be a field and d a positive integer. Let

$$k[T_1, \cdots, T_n]_{\leqslant d} := \{f \in k[T_1, \cdots, T_n] \colon \deg(f) \leqslant d\}$$

and let

$$k[T_0, T_1, \cdots, T_n]_d := \left\{ \sum_{m_0+\cdots+m_n=d} a_{(m_0,\cdots,m_n)} T_0^{m_0} \cdots T_n^{m_n} \colon a_{(m_0,\cdots,m_n)} \in k \right\}$$

be the space of homogeneous polynomials of degree d in $T_0, \ldots, T_n$. Show that the map

$$\Phi_d \colon k[T_1, \cdots, T_n]_{\leqslant d} \to k[T_0, T_1, \cdots, T_n]_d, \quad f \mapsto T_0^d f(\frac{T_1}{T_0}, \cdots, \frac{T_n}{T_0})$$

is an isomorphism of k-vector spaces.

Problem 7 Let k be a field. Show that the map

$$\Lambda \colon \{\text{lines in } \mathbb{A}^2(k)\} \to \{\text{lines in } \mathbb{P}^2(k)\}\backslash\{V_+(T_0)\}, \quad L = V(f) \mapsto V_+(\Phi_1(f))$$

is bijective, where Φ_1 is the map from Problem 6 for $n = 2$ and $d = 1$. (A line in $\mathbb{A}^2(k)$ is not required to pass through the origin.)

Problem 8 Let k be a field.

(1) (**Euler's identity**) Let d be a positive integer and $f \in k[T_0, T_1, \cdots, T_n]_d$. Show that

$$d \cdot f = \sum_{i=0}^{n} T_i \frac{\partial f}{\partial T_i}.$$

(2) Let $f \in k[T_1, T_2]_{\leqslant d}$ be non-constant and $F := \Phi_d(f)$.

(a) Show that $V(f) \subset \mathbb{A}^2(k)$ is mapped to $V_+(F) \subset \mathbb{P}^2(k)$ under the map

$$\Theta \colon \mathbb{A}^2(k) \to \mathbb{P}^2(k), \quad (a_1, a_2) \mapsto [1 : a_1 : a_2].$$

(b) Let $P \in V(f)$. Show that $V(f)$ is smooth at P if and only if $V_+(F)$ is smooth at $\Theta(P)$, and if this is the case, then $T_P V(f)$ is mapped to $T_{\Theta(P)} V_+(F)$ under the map Λ from Problem 7.