Algebraic Geometry I — Prof. Dr. Ulrich Görtz
WS 2022/23 — Dr. Heer Zhao

Problem sheet 12

Due date: Jan. 17, 2022.

Problem 45

Let k be an algebraically closed field of characteristic $\neq 2$. Let $f \in k[X_0, \dots, X_n]$ be homogeneous of degree 2, $f \neq 0$. We call $V_+(f) \subseteq \mathbb{P}^n_k$ a *quadric*. Which of the following quadrics in $\mathbb{P}^2_k$ are isomorphic as k-schemes?

$$V_+(X_0^2 + X_1^2), \quad V_+(X_0^2 + X_1^2 + X_2^2), \quad V_+(X_0X_2 - X_1^2).$$

Show that $V_+(X_0X_2 - X_1^2) \cong \mathbb{P}^1_k$.

Problem 46

Let $f\colon X \to Y$ be a morphism of schemes. Let $Y = \bigcup_i V_i$ be a cover by affine open subschemes such that for all i, $f^{-1}(V_i)$ is quasi-compact. Show that f is a *quasi-compact* morphism, i.e., $f^{-1}(V)$ is quasi-compact for every quasi-compact open $V \subseteq Y$.

Problem 47

Given an example of a field k and an irreducible homogeneous polynomial $f \in k[X, Y, Z]$ such that $V_+(f)$ (a closed subscheme of $\mathbb{P}^2_k$) is not isomorphic to $\mathbb{P}^1_k$ as a k-scheme.

Hint. One possible way is to think about singularities. Another possible way is to think about the number of k-rational points.

Problem 48

Let k be an algebraically closed field.

(1) Show that there exists a unique radical ideal $I \subseteq k[T_0, T_1, T_2]$ such that for $Z := V(I) \subseteq \mathbb{A}^3_k$ we have

$$Z(k) = \{(t, t^2, t^3);\ t \in k\} \subset k^3 = \mathbb{A}^3_k(k).$$

Show that $Z \cong \mathbb{A}^1_k$.

(2) Compute the closure $\overline{Z}$ of Z in $\mathbb{P}^3_k$ (with respect to the embedding $\mathbb{A}^3_k = D_+(X_0) \subseteq \mathbb{P}^3_k$, where we use X_0, X_1, X_2, X_3 as homogeneous coordinates on $\mathbb{P}^3_k$), and give a homogeneous ideal $J \subseteq k[X_0, X_1, X_2, X_3]$ such that $V_+(J)$ has underlying topological space $\overline{Z}$.