Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2022/23 Dr. Heer Zhao

Problem sheet 11

Due date: Jan. 10, 2023.

Problem 41

Let k be a field. Let X be the scheme obtained by gluing two copies of $\mathbb{A}^1_k = \operatorname{Spec} k[T]$ along the open subset $U = D(T)$, with respect to the identity map $U \to U$ ("the affine line with the origin doubled"). Prove that X is not an affine scheme.

Problem 42

Let k be an algebraically closed field. Give an example of a morphism of schemes $\mathbb{A}^2_k \to \mathbb{P}^1_k$ which is surjective (i.e., the underlying map on topological spaces is surjective).

Hint. First look for a map $\mathbb{A}^2_k(k) = k^2 \to \mathbb{P}^1(k)$ which is "defined by polynomials", i.e., of the form $(x,y) \mapsto (f(x,y) \colon g(x,y))$ for polynomials $f, g \in k[X,Y]$, and which is surjective.

Problem 43

Let R be a ring, $n \geqslant 1$. Let $B = R[X_0, \dots, X_n]$. Let $Z = V(X_0, \dots, X_n) \subseteq \mathbb{A}^{n+1}_R = \operatorname{Spec} B$, and let $U = \mathbb{A}^{n+1}_R \smallsetminus Z$, an open subscheme of $\mathbb{A}^{n+1}_R$.

Show that there is a "natural" morphism $p \colon U \to \mathbb{P}^n_R$ of R-schemes such that for every field k the induced map $U(k) \to \mathbb{P}^n_R(k)$ is given by $(x_0, \dots, x_n) \mapsto (x_0 \colon \cdots \colon x_n)$.

Hint. Let $\mathbb{P}^n_R = \bigcup_{i=0}^n U_i$ be the standard affine cover. Define morphisms $D(X_i) \to U_i$ and construct p by gluing of morphisms, applied to the compositions $D(X_i) \to U_i \to \mathbb{P}^n_R$.

Problem 44

We continue to work in the setting of Problem 43. Let $A = (a_{ij})_{i,j} \in GL_{n+1}(R)$ be an invertible $(n+1) \times (n+1)$-matrix with entries in R.

(1) The ring isomorphism
$$B \to B, \quad X_i \mapsto \sum_j a_{ij} X_j,$$
induces an isomorphism $\mathbb{A}^{n+1}_R \to \mathbb{A}^{n+1}_R$ of R-schemes.

Show that A restricts to an automorphism f_A of U.

(2) Show that there exists a unique automorphism f_A of $\mathbb{P}^n_R$ which fits into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f_A} & U \\ \downarrow & & \downarrow \\ \mathbb{P}^n_R & \xrightarrow{f_A} & \mathbb{P}^n_R. \end{array}$$

In this way we obtain a group homomorphism from $GL_{n+1}(R)$ into the group $\mathrm{Aut}_R(\mathbb{P}^n_R)$ of automorphisms of the R-scheme $\mathbb{P}^n_R$.

(3) Now let k be a field, $n = 1$. Let $\mathbb{P}^1_k = U_0 \cup U_1$ be the standard affine open cover. We have $\mathbb{P}^1(k) = U_0(k) \cup \{(0:1)\} = k \cup \{\infty\}$. Let $x, y, z \in \mathbb{P}^1(k)$ be distinct points. Show that there exists a unique automorphism f of $\mathbb{P}^1_k$ such that f is of the form f_A and such that

$$f(0) = x, \quad f(1) = y, \quad f(\infty) = z.$$