Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2022/23 Dr. Heer Zhao

Problem sheet 1

Due date: Oct. 18, 2022.

Problem 1

A non-empty topological space X is called *irreducible*, if it is not equal to the union of two proper closed subsets.

(1) Determine all topological spaces which are Hausdorff and irreducible. (Recall that a topological space X is called *Hausdorff*, if for any two points $u, v \in X$, $u \neq v$, there exist disjoint open subsets $U, V \subseteq X$ with $u \in U$, $v \in V$.

(2) Let X be a non-empty topological space. Prove that the following properties are equivalent:

  (i) The space X is irreducible.

  (ii) Every non-empty open subset $U \subseteq X$ is dense in X (i.e., the smallest closed subset of X containing U is X).

  (iii) Every non-empty open subset $U \subseteq X$ is connected. (A topological space is called *connected*, if it is non-empty and cannot be written as the union of two disjoint proper closed subsets.)

  (iv) Any two non-empty open subsets of X have non-empty intersection.

Problem 2

Let k be an infinite field.

(1) Let $n \geqslant 1$ and let $f \in k[T_1, \dots, T_n]$ be a polynomial such that $f(t_1, \dots, t_n) = 0$ for all $t_1, \dots, t_n \in k$. Prove that $f = 0$. *Hint.* You can use induction on n.

(2) Prove that k^n (with the Zariski topology) is irreducible. *Hint.* First show that if $Z \subsetneq k^n$ is a proper closed subset, then there exists $f \in k[T_1, \dots, T_n]$ such that $Z \subseteq V(f) \subsetneq k^n$.

Problem 3

Let k be a field, let $n, m \geqslant 0$, and let $f_1, \dots, f_n \in k[T_1, \dots, T_m]$ be polynomials. We consider k^m and k^n as topological spaces wit respect to the Zariski topology. Prove that the map

$$F\colon k^m \longrightarrow k^n, \quad (t_1, \dots, t_m) \mapsto (f_1(t_1, \dots, t_m), \dots, f_n(t_1, \dots, t_m)),$$

is continuous, i.e., for every closed subset $Z \subseteq k^n$, the inverse image $F^{-1}(Z)$ is closed.

Problem 4

Let k be an algebraically closed field, and let $d \geqslant 1$. We identify the set of all monic polynomials $f(X) = X^d + t_{d-1}X^{d-1} + \cdots + t_0$ of degree d with k^d by mapping f to $(t_0, \dots, t_{d-1})$.

Let $d = 2$. Prove that the subset of k^d corresponding to those polynomials which have a multiple zero is of the form $V(D)$ for a polynomial $D \in k[T_0, \dots, T_{d-1}]$.

Remark. The same result holds for all $d \geqslant 1$, but is more difficult to prove for $d > 2$. One way to do it is roughly as follows: View $f = X^d + t_{d-1}X^{d-1} + \cdots + t_0$ as a polynomial with coefficients in the field $K = k(t_0, \dots, t_{d-1})$ of rational functions in d variables over k. Let L be the splitting field of f, a Galois extension of K. Let α_i be the zeros of f in L, and let $D = \prod_{i<j}(\alpha_i - \alpha_j)^2$. Then use Galois theory to show that $D \in K$, and use that $k[t_0, \dots, t_{d-1}]$ is integrally closed to conclude that $D \in k[t_0, \dots, t_{d-1}]$. Alternatively, use the main theorem on elementary polynomials.