II The spectrum of a ring

26.10.2022

Reference: [GW] (2.1) – (2.4).

Görtz, Wedhorn, Algebraic Geometry I, see Moodle page

II.1 Grothendieck's approach to algebraic geometry in a nutshell

- <u>Classical approach</u>

Hilbert's Nullstellensatz: k alg. closed field

$$\mathrm{Max}\left(k[T_1,\dots,T_n]\right) \xleftrightarrow{1:1} k^n$$

$$(T_1-t_1,\dots,T_n-t_n) \longleftarrow (t_\bullet).$$

set of all maximal ideals

Given $f_1 \dots f_m \in k[T_1,\dots,T_n]$, the above bijection restricts to a bijection

$$\mathrm{Max}\left(k[T_1,\dots,T_n]/(f_1,\dots,f_m)\right) \xleftrightarrow{1:1} V(f_1,\dots,f_m)$$

<u>Proof</u>. • $\mathrm{Max}\left(k[T_\bullet]/\mathfrak{a}\right) \xleftrightarrow{1:1} \left\{ \mathfrak{m} \in \mathrm{Max}(k[T_\bullet]) \,;\, \mathfrak{a} \subseteq \mathfrak{m} \right\}$

• $f \in (T_1-t_1,\dots,T_n-t_n) \iff f(t_1,\dots,t_n) = 0$, in other words, $(T_1-t_1,\dots,T_n-t_n) \overset{\circledast}{=} \mathrm{Ker}\,\Phi$, where $\Phi : k[T_1,\dots,T_n] \to k$, $T_i \mapsto t_i$, is the evaluation homomorphism.

For ⊛, "$\subseteq$" is clear.

For the inclusion "$\supseteq$" one can either argue "directly" (e.g. using polynomial division), or use that $(T_1 - t_1, \dots, T_n - t_n)$ is a maximal ideal.

⟶ try to understand properties of $V(f_1, \dots, f_m)$ by studying the ring $k[T_1, \dots, T_n]/(f_1, \dots, f_m)$

<u>Problems</u>

- situation is more complicated, if k not algebraically closed
- in general, set of maximal ideals of a ring not well behaved (e.g. for ring homom $R \xrightarrow{\varphi} S$, $\mathfrak{m} \subset S$ max'l, $\varphi^{-1}(\mathfrak{m})$ not max in general)

- Grothendieck: attach, to any commutative ring R, the set $\operatorname{Spec} R$ of all <u>prime</u> ideals in R.

Spec R becomes a topological space when equipped with Zariski topology (Algebra 2/ see below)

$\rightsquigarrow$ functor $\underbrace{(\text{Rings})}_{\text{cat. of comm. rings}}^{op} \longrightarrow \text{Top}$ — category of topological spaces

Turns out: to fully capture the geometric aspects of the situation, one should (and can) add further structure and equip Spec R with what is called the structure of a locally ringed space (see below).

Then we will also be able to construct more general objects (such as projective space) by "gluing".

II.2 Reminder: the prime spectrum of a ring

<u>Def</u> (1) For a ring (always: with unit, commutative) A, let $\operatorname{Spec} A := \{\mathfrak{p} \subset A \text{ prime ideal}\}$

(2) We equip $\operatorname{Spec} A$ with the <u>Zariski topology</u> whose closed sets, by definition, are the sets of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A;\ \mathfrak{a} \subseteq \mathfrak{p}\},\quad \mathfrak{a} \subseteq A \text{ an ideal.}$$

NB. There is a small conflict of notation here with regard to the $V(\mathfrak{a})$ of the introduction.

(3) If $\varphi: A \to B$ is a ring homomorphism, then we define

$$\varphi^a : \operatorname{Spec} B \to \operatorname{Spec} A$$
$$\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q}).$$

<u>Lemma</u>. The map φ^a in Part (3) of the definition is continuous. We obtain a contravariant functor

$$(\mathrm{Rings})^{op} \to (\mathrm{Top}),$$
$$A \mapsto \operatorname{Spec} A.$$

category of topol. spaces

Example (0) For a ring A, $\operatorname{Spec} A = \emptyset \iff A = 0$

(1) $\operatorname{Spec} \mathbb{Z} = \{(0)\} \cup \{(p)\,;\ p \in \mathbb{Z} \text{ prime number}\}$

$\operatorname{Spec} k[X] = \{(0)\} \cup \{(f)\,;\ f \in k[X] \text{ monic, irreducible}\}$

k a field

(similarly for every principal ideal domain).

(2) k algebraically closed,

$$A = k[X_1, \dots, X_n] / (f_1, \dots, f_m)$$

set of max'l ideals of A

$$\operatorname{Spec} A = \operatorname{Max}(A) \ \dot\cup\ \underbrace{\{\mathfrak{p} \subset A \text{ prime ideal, not maximal}\}}$$

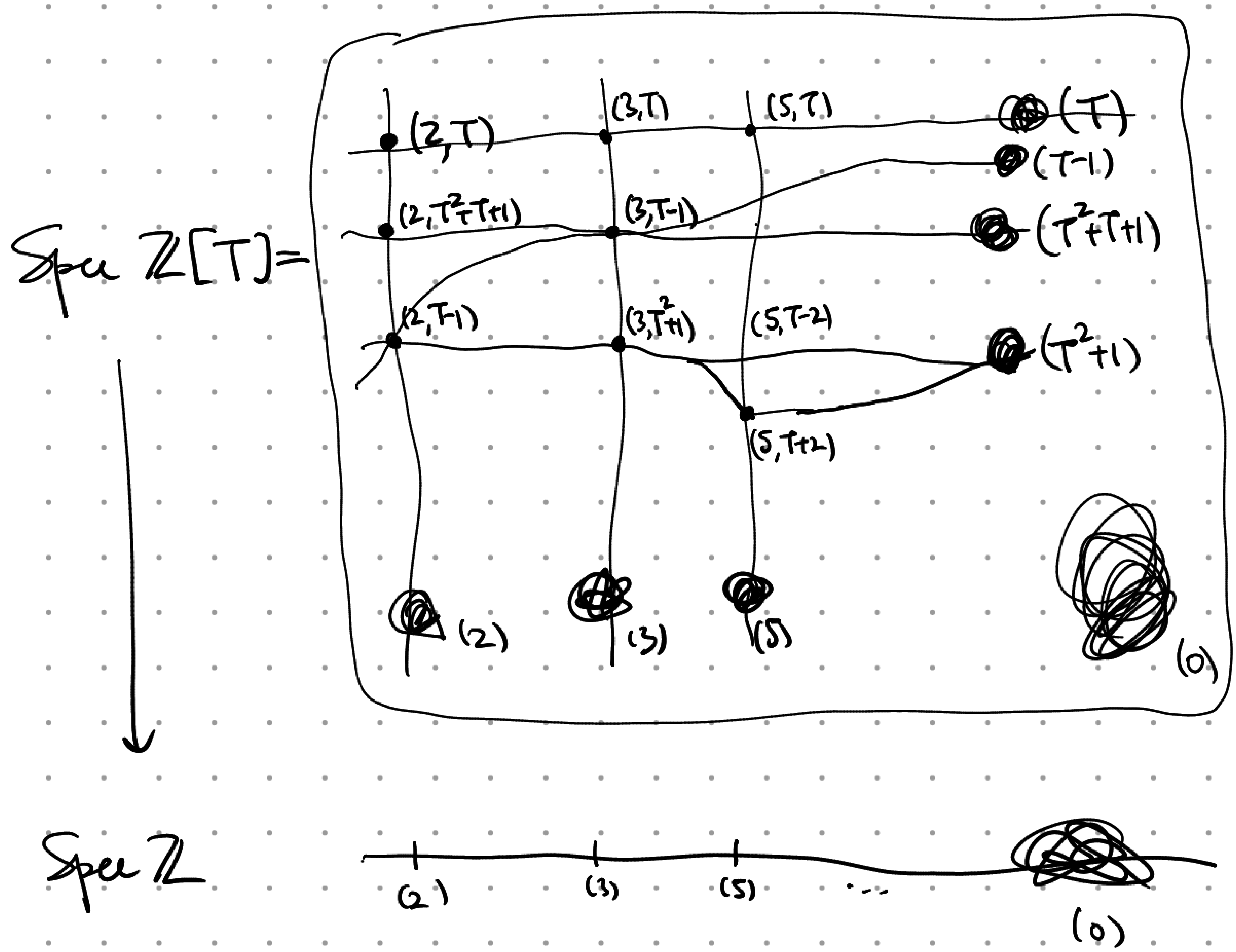
$\updownarrow$ 1:1

"classical algebraic geometry" $\longleftrightarrow$ $\{(x_i)_i \in k^n\,;\ \forall j : f_j(x_i) = 0\}$

one of our next goals: understand this part better

(3) Spec $\mathbb{Z}[T]$ (cf also

Mumford, Red book of varieties and schemes, II.1)



Two important classes of morphisms:

(1) A ring, $\mathfrak{a} \subseteq A$ ideal, $B = A/\mathfrak{a}$,

$\pi : A \to A/\mathfrak{a}$ canonical projection

$$\rightsquigarrow \quad \operatorname{Spec} A/\mathfrak{a} \longrightarrow \operatorname{Spec} A$$

$$\begin{tikzcd} \operatorname{Spec} A/\mathfrak{a} \arrow[r] \arrow[dr, "\sim"] & \operatorname{Spec} A \\ & V(\mathfrak{a}) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

homeomorphism
(i.e., isomorphism
of topological spaces)

(2) A ring, $f \in A$, A_f localization,

$\tau : A \to A_f$, $a \mapsto \frac{a}{1}$, natural map

$$\begin{tikzcd} \operatorname{Spec} A_f \arrow[r] \arrow[dr, "\sim"] & \operatorname{Spec} A \\ & D(f) := \operatorname{Spec}(A) \setminus V(f) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

homeomorphism

(The maps are continuous (by above general result) with the given image ("Commutative Algebra"). To show that the inverse map is also continuous, it is enough to show that the original map maps closed sets to closed sets (in $V(\mathfrak{a})$, $D(f)$, resp.)

(1) $\pi^a(V(\bar{\mathfrak{b}})) = V(\pi^{-1}(\bar{\mathfrak{b}}))$ (2) $\tau^a(V(\mathfrak{b})) = V(\tau^{-1}(\mathfrak{b})) \cap D(f)$.)

II.2 Closed subsets and radical ideals.

2.11.2022

Easy to see: For a ring A and ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$,

$V(\mathfrak{a}) = V(\mathfrak{b})$ does not imply $\mathfrak{a} = \mathfrak{b}$, in general.

The following proposition clarifies the situation.

For a subset $Y \subseteq \operatorname{Spec} A$, write $I(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$.

Prop Let A be a ring. The maps

$$\left\{ \begin{array}{c} \text{ideals } \mathfrak{a} \subset A \\ \text{s.t. } \mathfrak{a} = \operatorname{rad}(\mathfrak{a}) \end{array} \right\} \underset{I(Y) \leftarrow\!\shortmid Y}{\overset{\mathfrak{a} \mapsto V(\mathfrak{a})}{\rightleftarrows}} \left\{ \begin{array}{c} Y \subseteq \operatorname{Spec} A \\ \text{closed} \end{array} \right\}$$

"radical ideals"

are inverse to each other, and are in particular bijective. Both maps are inclusion-reversing.

(Recall: For $\mathfrak{a} \subseteq A$, $\operatorname{rad}(\mathfrak{a}) = \{ f \in A;\ \exists n \geq 1 : f^n \in \mathfrak{a} \}$

$= I(V(\mathfrak{a}))$ (*) the radical of $\mathfrak{a}$)

see "Commutative Algebra"

<u>Proof</u> • For any $Y \subseteq \operatorname{Spec} A$, $I(Y)$ is a radical ideal (clearly prime ideals are radical ideals, and intersections of radical ideals are radical ideals)

• For any ideal $\mathfrak{a} \subseteq A$, $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$ closed by def'n

Remains to show: the two maps are inverse to each other

• $I(V(\mathfrak{a})) = \operatorname{rad}(\mathfrak{a})$ by $(*)$

• For any $Y \subseteq \operatorname{Spec} A$, $V(I(Y)) = \overline{Y}$, the closure of Y in $\operatorname{Spec} A$:

"$\supseteq$" For $\mathfrak{p} \in Y$, $I(Y) \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(I(Y))$.

Hence $Y \subseteq V(I(Y))$, and since $V(I(Y))$ is closed, this implies $\overline{Y} \subseteq V(I(Y))$.

$V(I(Y)) \subseteq \overline{Y}$

'$\subseteq$'. We have $\overline{Y} = \bigcap_{\substack{Z \subseteq \operatorname{Spec} A \text{ closed} \\ Y \subseteq Z}} Z = \bigcap_{\substack{\mathfrak{a} \subseteq A \\ Y \subseteq V(\mathfrak{a})}} V(\mathfrak{a})$

$\rightsquigarrow$ enough to show: $Y \subseteq V(\mathfrak{a}) \Rightarrow V(I(Y)) \subseteq V(\mathfrak{a})$

But $Y \subseteq V(\mathfrak{a}) \Rightarrow \operatorname{rad}(\mathfrak{a}) = I(V(\mathfrak{a})) \subseteq I(Y)$

$\Rightarrow V(I(Y)) \subseteq V(\operatorname{rad}(\mathfrak{a})) = V(\mathfrak{a})$

$I(-)$ inclusion-reversing

$V(-)$ inclusion-reversing.

In particular: For $\mathfrak{p} \in \operatorname{Spec} A$, $Y := \{\mathfrak{p}\}$:

$$\overline{\{\mathfrak{p}\}} = V(I(\{\mathfrak{p}\})) = V(\mathfrak{p})$$

Remark k a field, A a finitely generated k-algebra (i.e. $A \cong k[X_1, \dots, X_n]/(f_1, \dots, f_m)$)

Commutative Algebra (general version of Hilbert's Nullstellensatz)

$\longrightarrow$ A is a Jacobson ring, i.e.

for all ideals $\mathfrak{a} \subseteq A$, $\operatorname{rad}(\mathfrak{a}) = \bigcap_{\substack{\mathfrak{m} \in \operatorname{Max} A \\ \mathfrak{a} \subseteq \mathfrak{m}}} \mathfrak{m}$

$\longrightarrow$ have bijections, inverse to each other:

$$
\begin{tikzcd}
\left\{ \begin{array}{c} \text{radical ideals} \\ \mathfrak{a} \subseteq A \end{array} \right\} \arrow[rr, shift left, "\mathfrak{a} \mapsto V(\mathfrak{a})"] \arrow[dr, shift left, "\mathfrak{a} \mapsto \boxed{V(\mathfrak{a}) \cap \operatorname{Max} A}"] & & \left\{ \begin{array}{c} Y \subseteq \operatorname{Spec} A \\ \text{closed} \end{array} \right\} \arrow[ll, shift left, "I(Y) \leftarrow Y"] \arrow[dl, shift left, "Y \mapsto Y \cap \operatorname{Max} A"] \\
& \left\{ \begin{array}{c} Z \subseteq \operatorname{Max} A \\ \text{closed} \end{array} \right\} \arrow[ul, shift left, "I(Z) \leftarrow Z"] \arrow[ur, shift left, "Z \mapsto \overline{Z}"] &
\end{tikzcd}
$$

for k alg. closed and $A = k[X_1, \dots, X_n]$, under the bijection $\operatorname{Max} A \longleftrightarrow k^n$, this set is the set that we denoted by $V(\mathfrak{a})$ in the Introduction.

II.3 Properties of the topological space Spec A

Let A be a ring. Recall sets $D(f) \subseteq \operatorname{Spec} A$:

Def. For $f \in A$, let $D(f) := \operatorname{Spec} A \setminus V(f)$

$$= \{ \mathfrak{p} \in \operatorname{Spec} A \,;\, f \notin \mathfrak{p} \},$$

an open subset of Spec A.

Open subsets of Spec A of this form are called <u>principal open subsets.</u>

Def Let X be a topological space.

A family $(U_i)_{i \in I}$ of open subsets of X is called a <u>basis of the topology of X</u>, if every open subset of X can be expressed as a union $\bigcup_{i \in J} U_i$ for some $J \subseteq I$.

<u>Example</u> $X = \mathbb{R}$ with the usual topology.

The family of bounded open intervals is a basis of the topology.

<u>Lemma</u> Let A be a ring. The family

$$(D(f))_{f \in A}$$ is a basis of the topology of $\operatorname{Spec} A$.

<u>Proof</u> Let $U \subseteq \operatorname{Spec} A$ open, say $U = \operatorname{Spec} A \setminus V(M)$.

Then $$U = \operatorname{Spec} A \setminus \left(\bigcap_{f \in M} V(f) \right) = \bigcup_{f \in M} D(f).$$

Some further properties:

$$D(1) = \operatorname{Spec} A, \quad D(0) = \emptyset, \quad D(f) \cap D(g) = D(fg).$$

<u>Prop</u> Let A be a ring, $f \in A$.

The set $D(f)$ (with the subspace topology for the inclusion $D(f) \subseteq \operatorname{Spec} A$) is quasi-compact (i.e. for every covering

$$D(f) = \bigcup_{i \in I} U_i$$ by open subsets $U_i \subseteq D(f)$,

there exists a <u>finite</u> set $J \subseteq I$ with $D(f) = \bigcup_{i \in J} U_i$).

<u>Proof</u>. Exercise.

8.11.2022

Recall the notion of irreducible topological space
(Problem sheet 1)

Def A topol. space $X \neq \emptyset$ is called irreducible,
if the following equivalent conditions are satisfied:

(i) If $A, B \subseteq X$ are closed with $X = A \cup B$, then $A = X$ or $B = X$.

(ii) Every non-empty open $U \subseteq X$ is dense in X, i.e., has closure $\overline{U} = X$.

(iii) Every non-empty open $U \subseteq X$ is connected, i.e. cannot be written as a disjoint union of proper closed subsets.

(iv) Any two non-empty open subsets of X have non-empty intersection.

Example (1) X irred., Hausdorff $\Rightarrow X = \{*\}$
a single point

(2) k alg. closed, Zariski topology

$V(X(X-1)) \subset \mathbb{A}^2(k)$ not connected
($\Rightarrow$ not irreducible)

$V(XY) \subset \mathbb{A}^2(k)$ connected, not irreducible

$V(X) \subset \mathbb{A}^2(k)$ irreducible ($\Rightarrow$ connected)

(3) Let X be a topological space s.t. there exists a point $\eta \in X$ such that $X = \overline{\{\eta\}}$. Then X is irreducible (because every non-empty open $U \subseteq X$ contains η).

<u>Prop</u> Let A be a ring, $Y \subseteq \operatorname{Spec} A$ a subset.

Then Y irreducible (with the subspace topology)

$\iff$ $I(Y)$ is a prime ideal.

In this case, $\overline{Y} = \overline{\{I(Y)\}}$.

<u>Proof</u> Write $\mathfrak{p} := I(Y)$.

$\Rightarrow$ Let $f, g \in A$ with $fg \in \mathfrak{p}$,

then $Y \subseteq V(fg) = V(f) \cup V(g)$

$\xrightarrow[Y \text{ irred.}]{} \quad Y \subseteq V(f)$ or $Y \subseteq V(g)$,

hence $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

$\Leftarrow$ $\overline{Y} = V(I(Y)) = V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ irred.

$\Longrightarrow$ $\overline{Y}$ irreducible $\underset{\text{Lemma below}}{\Longrightarrow}$ Y irred.

Lemma Let X be a topological space,

$Y \subseteq X$ a subset (considered as a topol. space with the subspace topology)

Then Y irreducible $\iff$ $\overline{Y}$ irreducible

closure of Y in X

Proof. For $U \subseteq X$ open,

$$U \cap Y \neq \emptyset \iff U \cap \overline{Y} \neq \emptyset .$$

Now use criterion (iv) of the equivalent conditions characterizing irreducible topological spaces.

<u>Cor</u> Let A be a ring. There is a bijection

$$\begin{aligned} \operatorname{Spec} A &\xrightarrow{1:1} \left\{ \begin{array}{l} \text{irreducible closed} \\ \text{subsets of } \operatorname{Spec} A \end{array} \right\} \\ \mathfrak{p} &\longmapsto V(\mathfrak{p}) \\ I(Y) &\longleftarrow Y, \end{aligned}$$

and in particular:

$$\overline{\{x\}} = \{x\}$$

$$\operatorname{Max} A \longleftrightarrow \begin{array}{l} \text{closed points} \\ x \in \operatorname{Spec} A \end{array}$$

$\rightsquigarrow$ every irreducible closed subset Y of $\operatorname{Spec} A$ has a unique <u>generic point</u>, i.e. a point $\eta \in Y$ s.t. $Y = \overline{\{\eta\}}$

Next we will look at the geometric meaning of the <u>minimal</u> prime ideals.

<u>Def</u> Let X be a topological space.

The maximal irreducible subsets of X are called the <u>irreducible components</u> of X

<u>Remark</u> Let X be a topological space.

(1) The irred. components of X are closed (use $Y \subseteq X$ irred $\Rightarrow \bar{Y}$ irred).

(2) Every irreducible subset of X is contained in an irred. component of X (use that the union of an ascending chain of irred. subsets is irreducible + Zorn's lemma). In particular, X is equal to the union of all its irreducible components

$$\mathfrak{p} \longmapsto V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$$

since $V(-)$ is inclusion-reversing

$$\left\{ \text{prime ideals in } A \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{irred. closed} \\ \text{subsets} \subseteq \operatorname{Spec} A \end{array} \right\}$$

$$\cup\!\!\mid \qquad\qquad\qquad\qquad \cup\!\!\mid$$

$$\left\{ \text{minimal prime ideals} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{irred. comp of} \\ \operatorname{Spec} A \end{array} \right\}$$

In particular: $\operatorname{Spec} A$ irreducible $\iff$ there exists a unique minimal prime ideal in A

$\iff$ $\operatorname{nil}(A)$ is a prime ideal

What we have done so far:

A a ring $\rightsquigarrow$ $\operatorname{Spec} A$ topol space

<u>Q.</u> What about "~~[illegible]~~", i.e. can we recover A from $\operatorname{Spec} A$? NO!

Is there a good way to think of (elements of) A "geometrically" / in terms of $\operatorname{Spec} A$? YES:

Should think of elements of A as "functions on $\operatorname{Spec} A$" (k alg. cl., $A = k[T_1, \ldots, T_n]$, $\operatorname{Spec} A \approx k^n$, $f \in A \rightsquigarrow$ polynomial fct. on k^n)

$$f \in A \rightsquigarrow \operatorname{Spec} A \longrightarrow \boxed{}$$

target depends on $\mathfrak{p}$...

$$\mathfrak{p} \longmapsto f(\mathfrak{p}) \in \kappa(\mathfrak{p}) := \operatorname{Frac}(A/\mathfrak{p})$$

residue class field at $\mathfrak{p}$

image of f under $A \to A/\mathfrak{p} \subseteq \operatorname{Frac}(A/\mathfrak{p})$

With this notation / point of view:

in $\kappa(x)$

$$V(f) = \{ \mathfrak{p} \in \operatorname{Spec} A \,;\, f \in \mathfrak{p} \} = \{ x \in \operatorname{Spec} A \,;\, f(x) = 0 \}$$

"vanishing set of f"

$$D(f) = \{ x \in \operatorname{Spec} A \,;\, f(x) \neq 0 \}.$$