III Sheaves

9.11.2022

reference: [GW] Sections (2.5) ff.

Slogan: "(type of) geometry" is determined by the (type of) functions one considers:

topological spaces $\leftrightarrow$ continuous functions
differentiable manifolds $\leftrightarrow$ differentiable fcts
complex manifolds $\leftrightarrow$ holomorphic functions

(algebraic geometry $\leftrightarrow$ polynomials)

III.1 Presheaves and sheaves

X topological space

Def A presheaf $\mathcal{F}$ (of sets) on X is given by

- for every open $U \subseteq X$ a set $\mathcal{F}(U)$
- for every two open $V \subseteq U \subseteq X$ a map ("restriction map") $\mathcal{F}(U) \xrightarrow{res^U_V} \mathcal{F}(V)$

such that
- $res^U_U = id_{\mathcal{F}(U)}$ for all U
- $res^V_W \circ res^U_V = res^U_W$ for all $W \subseteq V \subseteq U$

(i.o.w.: a functor $Open(X)^{op} \longrightarrow (Sets)$,

where $Open(X)$ is the category with objects the open subsets of X, morphisms

$$Hom(U,V) = \begin{cases} \{*\} & \text{if } U \subseteq V \\ \emptyset & \text{otherwise} \end{cases}$$

(and the unique identity elements and composition))

Def Let $\mathcal{F}, \mathcal{G}$ be presheaves on a topological space X. A <u>morphism</u> $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ of presheaves is a family of maps $\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U)$, $U \subseteq X$ open, s.t. for all $V \subseteq U \subseteq X$ open, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ {\scriptstyle res_{\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle res_{\mathcal{G}}} \\ \mathcal{F}(V) & \xrightarrow[\varphi(V)]{} & \mathcal{G}(V) \end{array}$$

commutes.

Notation / Terminology:

- $s|_V := res^U_V(s)$, $\quad s \in \mathcal{F}(U)$
- the elements of $\mathcal{F}(U)$ are called the <u>sections</u> of $\mathcal{F}$ on U

Def A sheaf (of sets) on X is a presheaf $\mathcal{F}$ such that for every $U \subseteq X$ open and every cover $U = \bigcup_{i \in I} U_i$ by open subsets $U_i \subseteq U$,

(a) If $s, s' \in \mathcal{F}(U)$ s.t. $s_{|U_i} = s'_{|U_i}$ $\forall i$, then $s = s'$.

(b) If $s_i \in \mathcal{F}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j$, then there exists a (unique) $s \in \mathcal{F}(U)$ s.t. $s_i = s_{|U_i}$ $\forall i$.

In other words:

$$\mathcal{F}(U) \xrightarrow{\rho} \prod_i \mathcal{F}(U_i) \underset{\sigma'}{\overset{\sigma}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

$$s \longmapsto (s_{|U_i})_i \qquad (s_i)_i \longmapsto (s_i|_{U_i \cap U_j})_{i,j}$$
$$\qquad\qquad\qquad\qquad\qquad\quad \longmapsto (s_j|_{U_i \cap U_j})_{i,j}$$

is exact (or: an equalizer), i.e. ρ injective and $\mathrm{im}(\rho) = \{ x \in \prod \mathcal{F}(U_i) \,;\ \sigma(x) = \sigma'(x) \}$

In particular: $\mathcal{F}(\emptyset) = \{ * \}$ (a one-point set) for every sheaf $\mathcal{F}$.

Def Let $\mathcal{F}, \mathcal{G}$ be sheaves on X.
A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves is a morphism $\mathcal{F} \to \mathcal{G}$ of presheaves.

Similarly: sheaves of groups, ab. groups, rings, R-modules (R a ring).

Examples

(1) X, Y topological spaces, then
$\mathcal{F}(U) := \{ f : U \to Y \text{ continuous} \}$
(with restriction of functions as restriction maps)
defines a sheaf on X.

(2) X differentiable manifold (e.g. $X \subseteq \mathbb{R}^n$ open)
$\rightsquigarrow$ $\mathcal{F}(U) := \{ f : U \to \mathbb{R} \text{ differentiable} \}$ defines a sheaf of rings on X.

(3) X complex manifold (e.g. $X \subseteq \mathbb{C}^n$ open)

$\rightsquigarrow$ $\mathcal{F}(U) := \{ f : U \to \mathbb{C} \text{ holomorphic} \}$

defines a sheaf of rings on X

(4) $X = \mathbb{R}$ (with the usual topology),

$\mathcal{F}(U) := \{ f : U \to \mathbb{R} \text{ bounded (+ differentiable)} \}$

defines a presheaf on X which is not a sheaf.

<u>Rmk.</u> In general, the restriction maps of a sheaf are neither injective nor surjective.

(But there are interesting situations where they are, e.g. for X a complex manifold, $\mathcal{F}$ as in (3) above $\emptyset \neq V \subseteq U \subseteq X$ open and U connected, the identity theorem says that $\mathcal{F}(U) \hookrightarrow \mathcal{F}(V)$ is injective.)

On the other hand, for every sheaf $\mathcal{F}$ on a topol. space X, $U \subseteq X$ open, $U = U_1 \cup U_2$ open cover, $U_1 \cap U_2 = \emptyset$: $\mathcal{F}(U) \xrightarrow{\sim} \mathcal{F}(U_1) \times \mathcal{F}(U_2)$ isomorphism.

Goal: Define a sheaf on Spec A for every ring A.

(1) $\mathcal{O}_X(X) := A$ "functions" on X

(2) What to do for open $U \subseteq X$?

Not so clear in general, but for

$U = D(s)$, $s \in A$, we have homeomorphism

$U \cong \operatorname{Spec}(A_s)$, so it makes sense to set

$\mathcal{O}_X(D(s)) := A_s$

(<u>note</u>: have to check this is well-defined!)

Since the $D(s)$ are a basis of the topology of Spec A, we will see (below) that this is in fact enough to define a sheaf on Spec A.

Sheaves on a basis of the topology

15.11.2022

Let X be a topol space and let $\mathcal{B}$ be a basis of the topology on X.

<u>Def</u> (1) A <u>presheaf $\mathcal{F}$ on $\mathcal{B}$</u> is given by

- for every $U \in \mathcal{B}$, a set $\mathcal{F}(U)$
- for all $V, U \in \mathcal{B}$ with $V \subseteq U$, a restriction map $res^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$

s.t. $res^U_U = id_U$, $res^U_W = res^V_W \circ res^U_V$ for all $\underbrace{W \subseteq V \subseteq U}_{\text{all} \in \mathcal{B}}$

(2) A <u>sheaf on $\mathcal{B}$</u> is a presheaf on $\mathcal{B}$ s.t.

for all $U, U_i, U_{ijk} \in \mathcal{B}$ s.t.

$$U = \bigcup_i U_i, \qquad U_i \cap U_j = \bigcup_k U_{ijk},$$

the sequence

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j,k} \mathcal{F}(U_{ijk})$$

is exact.

With the obvious notion of morphism of (pre-) sheaves on $\mathcal{B}$, obtain the categories of presheaves and of sheaves on $\mathcal{B}$.

<u>Prop</u> For every sheaf $\mathcal{F}$ on X, the "restriction"

$$\mathcal{B} \ni U \mapsto \mathcal{F}(U)$$

of $\mathcal{F}$ to $\mathcal{B}$ is a sheaf on $\mathcal{B}$.

In this way, we obtain an equivalence of categories $\quad (\text{sheaves on } X) \longrightarrow (\text{sheaves on } \mathcal{B})$

<u>Proof</u> The idea of the proof is to construct a quasi-inverse functor as follows:

Given a sheaf $\mathcal{F}$ on $\mathcal{B}$, construct a sheaf $\tilde{\mathcal{F}}$ on X as follows: For $U \subseteq X$ open, $\tilde{\mathcal{F}}(U)$ is given by requiring that

$$\tilde{\mathcal{F}}(U) \to \prod_{\substack{V \in \mathcal{B} \\ V \subseteq U}} \mathcal{F}(V) \rightrightarrows \prod_{\substack{V, V', W \in \mathcal{B} \\ W \subseteq V \cap V'}} \mathcal{F}(W)$$

(with maps as in the def'n of sheaf) is exact. Further details omitted.

"Reminder" Equivalence of categories

Let $\mathcal{C}$, $\mathcal{D}$ be categories.

Def A functor $F: \mathcal{C} \to \mathcal{D}$ is called an <u>equivalence of categories</u>, if the following equivalent conditions are satisfied:

(i) There exists a functor $G: \mathcal{D} \to \mathcal{C}$ which is quasi-inverse to F
(i.e. $G \circ F \cong \mathrm{id}_{\mathcal{C}}$, $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ (isomorphisms of functors))

(ii) F is fully faithful (i.e. for all $C, C' \in \mathrm{Ob}\,\mathcal{C}$, the map
$$\mathrm{Hom}_{\mathcal{C}}(C, C') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$$
induced by F is bijective) and essentially surjective (i.e., for all $D \in \mathrm{Ob}\,\mathcal{D}$ there exists $C \in \mathrm{Ob}\,\mathcal{C}$ s.t. $F(C) \cong D$.

((i) $\Rightarrow$ (ii) is relatively easy, (ii) $\Rightarrow$ (i) is more involved)

Example k a field

$\mathcal{D}$ the category of finite-dim'l k-vector spaces

$\mathcal{C}$ the category with objects $\mathrm{Ob}\,\mathcal{C} = \mathbb{N}$,

morphisms $\mathrm{Hom}_{\mathcal{C}}(n,m) = \mathrm{Mat}_{m\times n}(k)$,

composition of morphisms = product of matrices.

The functor
$$\begin{aligned} \mathcal{C} &\longrightarrow \mathcal{D} \\ n &\longmapsto k^n \\ \mathrm{Hom}_{\mathcal{C}}(n,m) \ni A &\longmapsto (k^n \to k^m,\ v \mapsto Av) \end{aligned}$$

is an equivalence of categories.

(Note that (ii) is easy to check (given basic linear algebra), and a quasi-inverse $\mathcal{D} \to \mathcal{C}$ on objects is given by $V \mapsto \dim V$, but to construct a quasi-inverse on morphisms, one needs to choose a basis for every finite-dim'l k-v.s.)

The (pre-) structure sheaf on the spectrum of a ring

A a ring, $X = \operatorname{Spec} A$ (with Zariski topology)

Note: May have $D(f) = D(g)$ for $f \neq g$ ...

Lemma (1) For $f, g \in A$, we have $D(f) \subseteq D(g)$ if and only if $\frac{g}{1} \in A_f^\times$

In this case, obtain
$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A_f \\ & A_g \arrow[u] \end{tikzcd}$$
commutative (for $A \to A_f$, $A \to A_g$ the natural maps)

(2) If $f, g \in A$ s.t. $D(f) = D(g)$, then there is a (unique) isomorphism $A_f \xrightarrow{\sim} A_g$ of A-algebras.

Proof. (1) $D(f) \subseteq D(g) \iff V(g) \subseteq V(f)$

$\iff \operatorname{rad}((f)) \subseteq \operatorname{rad}((g))$

$\iff \exists n: f^n \in (g)$

$\iff \exists n \geq 1, h \in A: gh = f^n$

$\iff \frac{g}{1} \in (A_f)^\times$.

In view of the lemma, for $U \subseteq \operatorname{Spec} A$ principal open, the ring A_f is indep. of the choice of f with $U = D(f)$ (up to unique isomorphism).

If one wants to make this more canonical, we can write this (for $f \in A$) as

$$A_f \overset{\sim}{=} S^{-1}A \quad \text{(isom of } A\text{-algebras)}$$

with $S = \{g \in A;\ D(f) \subseteq D(g)\}$, so S depends only on $D(f)$, not on f.

In view of this discussion, obtain a presheaf $\mathcal{O}_X$ on the basis of the topology of $\operatorname{Spec} A$ given by principal open subsets by setting

$$\mathcal{O}_X(U) := S^{-1}A, \qquad S = \{g \in A;\ D(f) \subseteq D(g)\}$$

with the unique A-algebra homom. $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ for principal open subsets $V \subseteq U$ as in Part (1) of the above lemma.

Later: $\mathcal{O}_X$ satisfies the sheaf property, hence extends to sheaf on $\operatorname{Spec} A$.

Stalks

X topol space, $x \in X$,
$\mathcal{F}$ a presheaf on X

The "stalk of $\mathcal{F}$ at x" (defined below) "captures the information given by $\mathcal{F}$ "at x" (at least if $\mathcal{F}$ is a sheaf). In classical language, the image of a section of $\mathcal{F}$ (on some neighborhood of x) in the stalk at x is the "germ" of the corresp. function at x.

Def With notation as above, the <u>stalk</u> of $\mathcal{F}$ at x is

$$\mathcal{F}_x := \operatorname*{colim}_{U \ni x} \mathcal{F}(U),$$

where the colimit runs over the set of open nbhds of x, partially ordered by "<u>reversed</u> inclusion", i.e. $U \leq V :\Leftrightarrow V \subseteq U$, and the transition maps are the restriction maps of $\mathcal{F}$.

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Reminder on colimits: Given

I a partially ordered set

$(F_i)_{i \in I}$ a family of sets,

for all $i \leq j$ a map $\varphi_{ji} : F_i \to F_j$,

In comparison to the first version of the lecture notes I changed from φ_{ij} to φ_{ji} to stay consistent with the problem sheets and the book [AW].

then a set C together with maps $\psi_i : F_i \to C$

such that $\psi_i = \psi_j \circ \varphi_{ji}$ for all $i \leq j$,

is called a colimit of the "inductive system"

(F_i, φ_{ji}) if it satisfies the following

universal property:

For every set T with maps $\xi_i : F_i \to T$,

s.t. $\xi_i = \xi_j \circ \varphi_{ji}$ $\forall i \leq j$,

there exists a unique map $C \xrightarrow{\xi} T$

s.t. $\xi_i = \xi \circ \psi_i$ for all i.

$$
\begin{tikzcd}
C \arrow[rr, dashed, "\exists! \xi"] & & T \\
F_i \arrow[u, "\psi_i"] \arrow[r, "\varphi_{ji}"'] \arrow[urr, "\xi_i"] & F_j \arrow[ul, "\psi_j"] \arrow[ur, "\xi_j"'] &
\end{tikzcd}
$$

Example If all F_i are subsets of a set M, $i \leq j \iff F_i \subseteq F_j$ and then φ_{ji} is the inclusion, then $C = \bigcup F_i$ (with the incl. $F_i \hookrightarrow C$) is a colimit of the F_i.

Analogous definition: colimit in a category $\mathcal{C}$ (colimits of groups, abelian groups, rings, R-modules, ...)

Rmk (1) If a colimit exists, it is uniquely determined up to unique isomorphism,

notation $\operatorname{colim}_{i \in I} F_i$

(alternative terminology: inductive limit, direct limit, $\varinjlim_{i \in I} F_i$)

(2) In a general category, colimits need not exist, but in the categories of sets, groups, abelian groups, rings, R-modules (R a fixed ring), all colimits exist.

(3) The notion of colimit is functorial:

Given $(F_i)_i$, $(G_i)_i$ and maps (more generally: morphisms in underlying category)

$$\begin{tikzcd} F_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & G_i \arrow[d] \\ F_j \arrow[r] & G_j \end{tikzcd}, \quad \text{obtain} \quad \begin{tikzcd} \operatorname{colim} F_i \arrow[r] & \operatorname{colim} G_i \\ F_i \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & G_i \arrow[u] \end{tikzcd}$$

(4) "Dually" (i.e. by "reversing the arrows") one has the notion of limit of a projective system: F_i, $F_j \to F_i$, $i \leq j$

$\rightsquigarrow$ consider universal property

$$\begin{tikzcd}
L \arrow[d] \arrow[dr] & T \arrow[l, dashed, "\exists!"'] \arrow[dl] \arrow[d] \\
F_j \arrow[r] & F_i
\end{tikzcd}$$

(5) If I is a set with partial order $i \leq j \Leftrightarrow i = j$, then the universal property of the (co-) limit is just the universal property of the (co-) product indexed by I.

(6) We say that I is <u>filtered</u>, if for all $i, j \in I$ there ex. $k \in I$ with $i \leq k$, $j \leq k$. Under this assumptions, colimits are sometimes better behaved (e.g. colim of ab. gps with respect to filtered index sets is an exact functor)

(7) Construction of colim of sets, (ab.) groups, rings, R-modules for a <u>filtered</u> index set I:

Given (F_i, φ_{ji}) as above,

$$C = \Big(\bigsqcup_{i \in I} F_i \Big) \Big/ \sim$$

with the natural maps $F_i \to C$

disjt union

and where $\sim$ is the equivalence relation $x_i \sim x_j \iff \exists k \geq i, j : \varphi_{ki}(x_i) = \varphi_{kj}(x_j)$
$x_i \in F_i$, $x_j \in F_j$

is a colimit of the sets F_i.

If the F_i carry additional structure

(s.t. as a group structure, ...), and the φ_{ij} are homomorphisms

define a gp structure on C by

group str. on F_k

$$x_i \cdot x_j := \text{image of } \varphi_{ki}(x_i) \cdot \varphi_{kj}(x_j)$$

$x_i \in F_i,\ x_j \in F_j$ in C for some $k \geq i, j$

$\square$

<u>Example</u> $X = \mathbb{C}$ (or $\mathbb{C}^n$ ...)

$\mathcal{F} = \mathcal{O}_X$, $\mathcal{O}_X(U) = \{ f : U \to \mathbb{C}$ holomorphic $\}$,

($U \subseteq X$ open)

$x \in X \rightsquigarrow$ can identify $\mathcal{O}_{X,x}$

with $\{ f = \sum_{n \geq 0} a_n X^n$; f converges in some open nbhd of $x \}$

$\subseteq \mathbb{C}[\![X]\!]$

Identity theorem $\Rightarrow$ the canonical map

$\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$

is injective for every connected

open $U \subseteq X$ with $x \in U$.

<u>Example</u> Let A be a ring,

$$X = \operatorname{Spec} A, \qquad x = \mathfrak{p}_x \in X$$

Consider the (pre-)sheaf $\mathcal{O}_X$ on X that we constructed above, i.e. $\mathcal{O}_X(D(f)) = A_f$.

Let us compute the stalk of $\mathcal{O}_X$ at x:

$$\mathcal{O}_{X,x} = \operatorname*{colim}_{\substack{U \subseteq X \\ x \in U}} \mathcal{O}_X(U) = \operatorname*{colim}_{\substack{f \in A,\, D(f) \subseteq X \\ x \in D(f)}} \mathcal{O}_X(D(f))$$

for every open nbhd $U \subseteq X$ of x, there exists $f \in A$ s.t. $x \in D(f) \subseteq U$

$\rightsquigarrow$ the $D(f)$ form a "cofinal subsystem" of the inductive system of all open neighborhoods of x.

$$= \underbrace{\operatorname*{colim}_{f \in A \setminus \mathfrak{p}_x} A_f} = A_{\mathfrak{p}}$$

(In the special case where A is a domain, all A_f are subrings of $\operatorname{Frac}(A)$ and the colim is simply their union.)

Coming back to the general definition of stalk (X topol space, $\mathcal{F}$ presheaf on X, $x \in X$):

$$\mathcal{F}_x = \operatorname*{colim}_{\substack{U \subseteq X \text{ open} \\ x \in U}} \mathcal{F}(U),$$

the functoriality of colimits shows that every morphism $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ of presheaves on X induces a morphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ on the stalks.

By construction, for every open nbhd $x \in U \subseteq X$, we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}_x \arrow[r, "\varphi_x"] & \mathcal{G}_x \end{tikzcd}$$

when the vertical maps are the natural maps into the colimit.

Many properties of sheaves / morphisms of sheaves can be "detected on stalks", e.g.

Lemma Let X be a topol space, $\mathcal{F}, \mathcal{G}$ sheaves on X

(1) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

Then the following are equivalent:

(i) for all $U \subseteq X$ open, $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ injective

(ii) for all $x \in X$, $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ injective

(2) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

Then the following are equivalent:

(i) for all $U \subseteq X$ open, $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ bijective

(ii) for all $x \in X$, $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ bijective

EXERCISE

(3) Let $\varphi, \psi : \mathcal{F} \rightrightarrows \mathcal{G}$ be morphisms of sheaves.

Then $\varphi = \psi \iff \forall x \in X : \varphi_x = \psi_x$.

Proof

(1) (i) $\Rightarrow$ (ii) Let $s, s' \in \mathcal{F}_x$ s.t. $\varphi_x(s) = \varphi_x(s')$.

Represent s, s' by sections $s_U, s'_U \in \mathcal{F}(U)$ for some sufficiently small nbhd $U \ni x$

(i.e. $s_U \in \mathcal{F}(U)$ maps to s under the natural map $\mathcal{F}(U) \to \mathcal{F}_x$)

It follows that $\varphi(U)(s_U)$ and $\varphi(U)(s'_U)$ (elements of $\mathcal{G}(U)$) have the same image in $\mathcal{G}_x$.

Therefore there exists V, $x \in V \subseteq U$ s.t.

$$\varphi(V)(s_U|_V) = \varphi(U)(s_U)|_V = \varphi(U)(s'_U)|_V = \varphi(V)(s'_U|_V)$$

Since $\varphi(V)$ is injective by assumption, $s_U|_V = s'_U|_V$ and hence $s = s'$.

(ii) $\Rightarrow$ (i). Let $s, s' \in \mathcal{F}(U)$ with $\varphi(U)(s) = \varphi(U)(s')$.

Let $x \in U$.

Then the assumption on φ_x implies that s, s' have the same image in $\mathcal{F}_x$.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}_x \arrow[r, hook, "\varphi_x"'] & \mathcal{G}_x \end{tikzcd}$$

Therefore there exists an open nbhd $V_x \subseteq U$ of x s.t. $s|_{V_x} = s'|_{V_x}$.

Altogether we obtain an open cover $U = \bigcup_{x \in U} V_x$ of U s.t. $s|_{V_x} = s'|_{V_x}$ for all x, so the first sheaf property of $\mathcal{F}$ implies that $s = s'$.

<u>Definition</u> Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X.

(1) We call φ <u>injective</u>, if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ on stalks is injective

(2) We call φ <u>surjective</u>, if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ on stalks is surjective.

(3) We call φ <u>bijective</u>, if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ on stalks is bijective.

22.11.2022 (1)

The structure presheaf on Spec A (A a ring) is a sheaf

Recall: A a ring $\rightsquigarrow$ $\operatorname{Spec} A = X$

$D(f)$, $f \in A$, basis of topology

Structure presheaf $\mathcal{O}_X$ defined by $\mathcal{O}_X(D(f)) := A_f$
(+ natural restriction maps)

Prop The presheaf $\mathcal{O}_X$ on the basis $\mathcal{B}$ of principal open subsets of Spec A is a sheaf on $\mathcal{B}$
(and hence extends to a unique sheaf on $X = \operatorname{Spec} A$ again denoted by $\mathcal{O}_X$, called the structure sheaf on Spec A).

Proof. Need to show: for $f, f_i, f_i f_j \in A$,

$$D(f) = \bigcup_{i \in I} D(f_i),$$

the sequence

$$0 \to A_f \to \prod_i A_{f_i} \to \prod_{i,j} A_{f_i f_j}$$

$$s \mapsto \left(\frac{s}{1}\right)_i$$

$$(s_i) \mapsto \left(\frac{s_i}{1} - \frac{s_j}{1}\right)_{i,j}$$

is exact.

Reduction steps

- replacing A by A_f, we may assume $f = 1$ (and hence $A_f = A$).
- Since all the sets $D(g)$ are quasi-compact, we may assume that I is a finite set.

(I) Consider $0 \to A \xrightarrow{\rho} \prod_i A_{f_i} \xrightarrow{\sigma} \prod_{i,j} A_{f_i f_j}$

<u>ρ injective</u> The assumption $\operatorname{Spec} A = \bigcup D(f_i)$ implies (in fact, is equiv. to) that the f_i, $i \in I$, generate the unit ideal of A.

(*) Then for every $N \geq 0$, the powers f_i^N also generate the unit ideal (since $\operatorname{rad}(f_i^N) = \operatorname{rad}(f_i)$).

If $s \in A$ s.t. $\rho(s) = 0$, then

$\frac{s}{1} = 0 \in A_{f_i}$ for all i,

i.e. $f_i^{N_i} s = 0$ for some $N_i \geqslant 0$.

May assume that $N_i =: N$ is indep. of i.

Use (*) to write $1 = \sum g_i f_i^N$, $g_i \in A$.

Then $s = \left(\sum g_i f_i^N\right) s = 0$.

$\underline{im(\rho) = ker(\sigma)}$. "$\subseteq$" is clear (and actually holds for every presheaf)

"$\supseteq$": Let $(s_i)_i \in ker(\sigma)$.

We write $s_i = \frac{a_i}{f_i^N}$, $a_i \in A$, $N \geq 0$

Then $(s_i)_i \in ker(\sigma) \iff \frac{a_i}{f_i^N} = \frac{a_j}{f_j^N} \in A_{f_i f_j}$ $\forall i,j$

$\iff$ there ex. $M \geq 0$: $(f_i f_j)^M (a_i f_j^N - a_j f_i^N) = 0$ (**)

Use (*) above to write $1 = \sum g_i f_i^{M+N}$, $g_i \in A$.

Let $a = \sum_j g_j f_j^M a_j$

$\underline{\text{Claim}}$ $(s_i)_i = \rho(a)$.

$\underline{\text{Proof of claim.}}$ We need $\frac{a}{1} = \frac{a_i}{f_i^N} \in A_{f_i}$.

It is enough to show $f_i^M (a_i - a f_i^N) = 0$.

In fact, we have

$$a f_i^{M+N} = \sum_j g_j f_j^M f_i^{M+N} a_j \overset{(**)}{=} \sum_j g_j f_i^M f_j^{M+N} a_i = a_i f_i^M.$$

(II) (alternative proof using a little bit more commutative algebra)

To check that

$$0 \to A \to \prod A_{f_i} \to \prod A_{f_i f_j}$$

as above is exact, it is enough that the sequence induces an exact sequence after localization at every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$.

Now $(A_f)_{\mathfrak{p}} = A_{\mathfrak{p}}$ whenever $f \notin \mathfrak{p}$.

So localizing the above sequence at $\mathfrak{p}$, we obtain (using that the products are finite, hence commute with localization)

$$0 \to A_{\mathfrak{p}} \xrightarrow{\rho_{\mathfrak{p}}} \prod_{\substack{i \\ f_i \notin \mathfrak{p}}} A_{\mathfrak{p}} \times \prod_{\substack{j \\ f_j \in \mathfrak{p}}} (A_{f_j})_{\mathfrak{p}} \xrightarrow{\sigma_{\mathfrak{p}}} \prod_{\substack{i,j \\ f_i f_j \notin \mathfrak{p}}} A_{\mathfrak{p}} \times \prod_{\substack{i,j \\ f_i f_j \in \mathfrak{p}}} (A_{f_i f_j})_{\mathfrak{p}}$$

The condition $f_i \notin \mathfrak{p}$ is equivalent to $\mathfrak{p} \in D(f_i)$. Since $\operatorname{Spec} A = \bigcup D(f_i)$ by assumption, the product $\prod_{\substack{i \\ f_i \notin \mathfrak{p}}} A_\mathfrak{p}$ has non-empty index set.

It is then clear that the map $\rho_\mathfrak{p}$ is injective.

It remains to show that $\ker(\sigma_\mathfrak{p}) \subseteq \operatorname{im}(\rho_\mathfrak{p})$.

So let $((a_i)_i, (b_j)_j) \in \ker(\sigma_\mathfrak{p})$.

Then $a_i = a$ for some $a \in A_\mathfrak{p}$ for all i with $f_i \notin \mathfrak{p}$.

<u>Claim</u> $\rho_\mathfrak{p}(a) = ((a_i)_i, (b_j)_j))$

<u>Proof of claim</u> Let $j \in I$, $f_j \in \mathfrak{p}$. We need $a = b_j \in (A_{f_j})_\mathfrak{p}$

Let $i \in I$ with $f_i \in \mathfrak{p}$. Then $a - b_j = a_i - b_j \in \underbrace{(A_{f_i f_j})_\mathfrak{p}}_{\substack{\shortparallel \\ (A_{f_j})_\mathfrak{p},}}$

so the claim follows.

<u>Remark</u> Note that the localizations $A_\mathfrak{p}$ are precisely the stalks of the (pre-)sheaf $\mathcal{O}_{\operatorname{Spec} A}$.

Restriction of a sheaf to an open subspace

22.11.2022 (2)

Let X be a topol. space, $\mathcal{F}$ a (pre-)sheaf on X, $U \subseteq X$ open.

Then $\mathcal{F}_{|U}$ denotes the following (pre-) sheaf on U (regarded as a topol space with the subspace topology):

$$\mathcal{F}_{|U}(V) := \mathcal{F}(V), \qquad V \subseteq U \text{ open},$$

$$\mathrm{res}^V_W := \mathrm{res}^V_W \text{ (restriction for } \mathcal{F}) \qquad W \subseteq V \subseteq U \text{ open}.$$

The sheafification of a presheaf

X topological space, $\mathcal{F}$ presheaf on X.

Goal Attach to $\mathcal{F}$ a sheaf $\tilde{\mathcal{F}}$ which is as "close" to $\mathcal{F}$ as possible. More precisely, we will achieve the following:

- $\mathcal{F}$ and $\tilde{\mathcal{F}}$ have the same stalks
- the construction is functorial, there is a morphism $\mathcal{F} \to \tilde{\mathcal{F}}$ of presheaves, and $\tilde{\mathcal{F}}$ satisfies a universal property.

We use the universal property to define this "sheafification" of $\mathcal{F}$:

Def. For $X, \mathcal{F}$ as above, a sheaf $\tilde{\mathcal{F}}$ on X together with a morphism $\iota_{\mathcal{F}} : \mathcal{F} \to \tilde{\mathcal{F}}$ of presheaves is called a sheafification of $\mathcal{F}$ if for every sheaf $\mathcal{G}$ on X and morphism $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ of presheaves there exists a unique morphism $\psi : \tilde{\mathcal{F}} \to \mathcal{G}$ of (pre-)sheaves such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\iota_{\mathcal{F}}"'] & \mathcal{G} \\ \tilde{\mathcal{F}} \arrow[ur, dashed, "\exists!\, \psi"'] & \end{tikzcd}$$

From this definition, it is clear that a sheafification is unique (up to unique isomorphism) if it exists, so that we usually speak of the sheafification of $\mathcal{F}$.

Proposition Let $\mathcal{F}$ be a presheaf on a topol space X.

Then $\mathcal{F}$ has a sheafification $\tilde{\mathcal{F}}$.

For every $x \in X$, the morphism $\iota_{\mathcal{F}} : \mathcal{F} \to \tilde{\mathcal{F}}$ induces a bijection $\mathcal{F}_x \to \tilde{\mathcal{F}}_x$ on stalks.

For every morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves, there is a unique morphism $\tilde{\varphi} : \tilde{\mathcal{F}} \to \tilde{\mathcal{G}}$ of sheaves s.t. $\tilde{\varphi} \circ \iota_{\mathcal{F}} = \iota_{\mathcal{G}} \circ \varphi$.

Proof (sketch)

For $U \subseteq X$ open, we set

image of t under natural map $\mathcal{F}(W) \to \mathcal{F}_w$

$$\tilde{\mathcal{F}}(U) = \Big\{ (s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x \ ; \ \forall x \in U \ \exists\, x \in \underset{\text{open}}{W \subseteq U},\ t \in \mathcal{F}(W) : \forall w \in W : t_w = s_w \Big\}$$

and for $V \subseteq U \subseteq X$ open, define res^U_V as the restriction of the projection $\prod_{x \in U} \mathcal{F}_x \to \prod_{x \in V} \mathcal{F}_x$ to $\mathcal{F}(U)$.

One checks that this defines a sheaf $\tilde{\mathcal{F}}$ on X.

We obtain a morphism $\mathcal{F} \xrightarrow{\iota_{\mathcal{F}}} \tilde{\mathcal{F}}$ of presheaves by setting, for $U \subseteq X$ open:

$$\iota_{\mathcal{F}}(U) : \mathcal{F}(U) \longrightarrow \tilde{\mathcal{F}}(U).$$
$$s \longmapsto (s_x)_{x \in U}$$

One checks that $\iota_{\mathcal{F}}$ is a morphism of presheaves and induces isomorphisms on all stalks.

Functoriality and universal property:

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves

$\rightsquigarrow$ $\tilde{\mathcal{F}} \xrightarrow{\tilde{\varphi}} \tilde{\mathcal{G}}$ morph of presheaves given by

$$\tilde{\mathcal{F}}(U) \longrightarrow \tilde{\mathcal{G}}(U)$$
$$(s_x)_x \longmapsto (\varphi_x(s_x))_x$$

$\rightsquigarrow$
$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\ \tilde{\mathcal{F}} \arrow[r, "\tilde{\varphi}"] & \tilde{\mathcal{G}} \end{tikzcd}$$
Commutative

On stalks, $\tilde{\varphi}_x = \varphi_x$ is determined by φ, hence $\tilde{\varphi}$ is the unique morphism (of sheaves (!)) making the diagram commutative.

If $\mathcal{G}$ is a sheaf, then the morphism $\mathcal{G} \xrightarrow{\iota_{\mathcal{G}}} \tilde{\mathcal{G}}$ is an isomorphism (since it is bijective on stalks).

This shows the universal property of $(\tilde{\mathcal{F}}, \iota_{\mathcal{F}})$.

<u>Remark</u> If $\mathcal{F}$ is a presheaf of groups, abelian groups, rings, ..., then the sheafification $\tilde{\mathcal{F}}$ "is" a sheaf of groups, abelian groups, rings, ...

Example (constant sheaf)

Let X be a topological space, and let E be a set.

Let $\mathcal{F}$ be the presheaf $\mathcal{F}(U) = E$, $\quad \left(\underset{\text{open}}{V \subseteq U \subseteq X} \right)$,

$\mathrm{res}^U_V = \mathrm{id}_E$

The sheafification $\tilde{\mathcal{F}}$ of $\mathcal{F}$ is called the constant sheaf with value E on X and often denoted by $\underline{E}_X$.

One can show that $\underline{E}_X(U) = \left\{ \begin{array}{c} \text{locally constant} \\ \text{functions } U \to E \end{array} \right\}$

(with restriction of functions as the restriction maps).

(Similarly, E an (abelian) group, a ring, ...)

III.2 Direct and inverse image

In this section, we consider a continuous map $f: X \to Y$ between topological spaces X, Y. We want to study how we can "transport" sheaves from X to Y (and from Y to X) via f.

<u>Direct image</u> If $\mathcal{F}$ is a presheaf on X, we define the <u>direct image</u> (or <u>pushforward</u>) $f_* \mathcal{F}$ of $\mathcal{F}$ under f by

$$(f_* \mathcal{F})(V) := \mathcal{F}(f^{-1}(V)), \quad V \subseteq Y \text{ open}$$

$$\mathrm{res}^V_W = \mathrm{res}^{f^{-1}(V)}_{f^{-1}(W)} \qquad W \subseteq V \subseteq Y \text{ open}$$

One checks that if $\mathcal{F}$ is a sheaf, then $f_* \mathcal{F}$ is a sheaf on Y.

The construction is functorial, i.e. for every morphism $\mathcal{F} \to \mathcal{G}$ of presheaves on X we obtain a morphism $f_* \mathcal{F} \to f_* \mathcal{G}$, and this is compatible with composition.

Inverse image of a (pre-) sheaf

29.11.2022

Again, let $f: X \to Y$ be a continuous map.

For $\mathcal{G}$ a presheaf on Y, we define a presheaf

$$f^+\mathcal{G}: \quad U \mapsto \operatorname*{colim}_{\substack{V \subseteq Y \text{ open} \\ f(U) \subseteq V}} \mathcal{G}(V)$$

where the transition maps of the system $(\mathcal{G}(V))_V$ are the restriction maps, as usual.

One checks that for $U' \subseteq U \subseteq X$ open, we have a natural restriction map $(f^+\mathcal{G})(U) \to (f^+\mathcal{G})(U')$ (basically because $f(U') \subseteq f(U)$, so $V \supseteq f(U)$ implies $V \supseteq f(U')$). So we obtain a presheaf $f^+\mathcal{G}$ on X.

However, even if $\mathcal{G}$ is a sheaf, $f^+\mathcal{G}$ in general is not a sheaf on X.

Therefore we define the <u>inverse image</u> (or <u>pullback</u>) $f^{-1}\mathcal{G}$ of $\mathcal{G}$ under f as the sheafification of $f^+\mathcal{G}$.

One checks that the construction $\mathcal{F} \rightsquigarrow f^+\mathcal{F}$ is functorial in $\mathcal{F}$, so f^+ defines a functor between the category of sheaves on Y to the category of sheaves on X.
Since sheafification is a functor, an analogous statement holds for f^{-1}.

Similarly, one has the inverse image of (pre-)sheaves of abelian groups, or of rings, ...

Example X topol space, $\mathcal{F}$ presheaf on X, $x \in X$, $i : \{x\} \to X$ the inclusion

$\rightsquigarrow$ $i^{-1}\mathcal{F}$ = the sheaf on $\{x\}$ with sections $\mathcal{F}_x$ on $\{x\}$

Going back to the general setting ($f: X \to Y$, $\mathcal{G}$ on Y), we have, for $x \in X$,

$$(f^{-1}\mathcal{G})_x = (f^{+}\mathcal{G})_x = \operatorname*{colim}_{x \in U} f^{+}\mathcal{G}(U) =$$

$$\operatorname*{colim}_{\substack{x \in U \\ f(U) \subseteq V \subseteq Y}} \mathcal{G}(V) = \mathcal{G}_{f(x)}$$

f continuous
$\rightsquigarrow$ the V with $f(U) \subseteq V$ for some open nbhd U of x form a "basis of open nbhds of $f(x)$"

Now consider continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$.

It is easy to see that $(gf)^{+}\mathcal{H} = f^{+}(g^{+}\mathcal{H})$ for every presheaf $\mathcal{H}$ on Z. By the above, then the natural morphism $f^{-1}(g^{+}(\mathcal{H})) \longrightarrow f^{-1}(g^{-1}\mathcal{H})$ induces an isomorphism on all stalks, and hence is a sheaf isomorphism.

Hence

$$(g \circ f)^{-1}(\mathcal{H}) = \text{sheafif. of } (g \circ f)^{+}(\mathcal{H}) = \text{sheafif. of } f^{+}(g^{+}\mathcal{H}) = f^{-1}(g^{+}\mathcal{H}) \cong f^{-1}g^{-1}\mathcal{H},$$

i.e., "inverse image of sheaves is compatible with composition of continuous maps".

The following proposition states an important relation between direct and inverse image.

<u>Prop.</u> Let $f: X \to Y$ be a continuous map, let $\mathcal{F}$ be a sheaf on X and let $\mathcal{G}$ be a presheaf on Y. There is a bijection

$$\begin{aligned} \mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) &\longrightarrow \mathrm{Hom}_{PreSh(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ \varphi &\longmapsto \varphi^b \\ \psi^\# &\longleftarrow \psi \end{aligned}$$

which is functorial in $\mathcal{F}$ and in $\mathcal{G}$.

<u>Proof.</u> First note that $\mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}(f^{+}\mathcal{G}, \mathcal{F})$ by the definition of f^{-1} and the universal property of the sheafification.

We construct maps $\mathrm{Hom}(f^{+}\mathcal{G}, \mathcal{F}) \underset{\psi^\# \leftarrow \psi}{\overset{\varphi \mapsto \varphi^b}{\rightleftarrows}} \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$ which are inverse to each other as follows.

$$\boxed{\begin{array}{c} \varphi \longmapsto \varphi^{\flat} \\ \mathrm{Hom}(f^{+}\mathcal{G}, \mathcal{F}) \rightleftarrows \mathrm{Hom}(\mathcal{G}, f_{*}\mathcal{F}) \\ \psi^{\#} \longleftarrow\!\shortmid\ \psi \end{array}}$$

Given φ, define $\varphi^{\flat}$ on an open $V \subseteq Y$ by

$$\begin{tikzcd}
\mathcal{G}(V) \arrow[r] \arrow[dr] & (f_{*}\mathcal{F})(V) = \mathcal{F}(f^{-1}(V)) \\
& (f^{+}\mathcal{G})(f^{-1}(V)) \arrow[u, "\varphi(f^{-1}(V))"']
\end{tikzcd}$$

natural map into the colim, note $f(f^{-1}(V)) \subseteq V$

and given ψ, define $\psi^{\#}$ on an open $U \subseteq X$ by

$$\begin{tikzcd}
(f^{+}\mathcal{G})(U) \arrow[rr, dashed] & & \mathcal{F}(U) \\
\mathcal{G}(V) \arrow[u] \arrow[rr, "\psi(V)"'] & & \mathcal{F}(f^{-1}(V)) \arrow[u, "\mathrm{res}^{f^{-1}(V)}_{U}"']
\end{tikzcd}$$

via universal property of colim

$f(U) \subseteq V$

$(\Rightarrow U \subseteq f^{-1}(V))$

We omit checking that

- these are morph of presheaves (i.e., compatible with restriction maps)
- $-^{\flat}$ and $-^{\#}$ are inverse to each other, and
- are functorial in $\mathcal{F}$ and in $\mathcal{G}$.

In other words, the proposition states, that f^{-1}, f_* are a pair of adjoint functors.

Aside Adjoint functors

$\mathcal{C}$, $\mathcal{D}$ categories, consider functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$

Def. We say that (F, G) is a pair of adjoint functors, or that F is a left adjoint of G, or that G is a right adjoint of F (and sometimes write $F \dashv G$), if there are, for each $X \in \mathrm{Ob}\,\mathcal{C}$, $Y \in \mathrm{Ob}\,\mathcal{D}$, isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, GY) \quad (*)$$

that are functorial in X and in Y.

If one of the two functors F, G is fixed, then $(*)$ can be expressed as a universal property for the other one. Hence, if a (left or right) adjoint functor exists, then it is uniquely determined up to unique isom.

Given a pair (F, G) of adjoint functors, we obtain (using $(*)$ for identity morphisms) for every $X \in \mathrm{Ob}\,\mathcal{C}$, $Y \in \mathrm{Ob}\,\mathcal{D}$ morphisms

$Y = FX$ $\rightsquigarrow$ $X \xrightarrow{\eta_X} GFX$,

$X = GY$ $\quad FGY \xrightarrow[\varepsilon_Y]{} Y$

s.t. for all morphisms $FX \xrightarrow{f} Y$, $X \xrightarrow{g} GY$ corresp. to each other via $(*)$ the foll triangles commute

$$\begin{tikzcd} FX \arrow[rr, "f"] \arrow[dr, "F(g)"'] & & Y \\ & FGY \arrow[ur, "\varepsilon_Y"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\eta_X"'] & & GY \\ & GFX \arrow[ur, "G(f)"'] & \end{tikzcd}$$

We can view the family η_X of morphisms in $\mathcal{C}$ (for all $X \in \mathrm{Ob}\,\mathcal{C}$) as a morphism $\mathrm{id}_{\mathcal{C}} \xrightarrow{\eta} GF$ of functors $\mathcal{C} \to \mathcal{C}$ and call η the <u>unit</u> of the adjunction. Similarly, we have the <u>counit</u> $\varepsilon : FG \to \mathrm{id}_{\mathcal{D}}$ of the adjunction.

30.11.2022

<u>Examples</u> • inverse image and direct image of sheaves under a continuous map:

$f^{-1} \dashv f_*$ (similarly, for presheaves: $f^+ \dashv f_*$)

• sheafification $\mathrm{PreSh}(X) \underset{\iota}{\overset{\sim}{\rightleftarrows}} \mathrm{Sh}(X)$

"inclusion", i.e. view a sheaf as a presheaf

have seen (for $\mathcal{F}$ presheaf, $\mathcal{G}$ sheaf on X):

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\tilde{\mathcal{F}}, \mathcal{G}) = \mathrm{Hom}_{\mathrm{PreSh}(X)}(\mathcal{F}, \iota(\mathcal{G}))$$

⇝ sheafification left adjoint to the inclusion ι.

• free objects / forgetful functors

– Let k be a field. Consider the forgetful functor $(\mathrm{Vect}_k) \xrightarrow{G} (\mathrm{Sets})$ from the category of k-vector spaces to the category of sets (mapping a vector space V to the underlying set, i.e. forgetting the addition and mult. by scalars).

A left adjoint of G is given by the functor
$F : (\mathrm{Sets}) \to (\mathrm{Vect}_k)$, $I \mapsto k^{(I)} := \bigoplus_{i \in I} k$,

(the "standard k-vector space" with basis e_i, $i \in I$)

because
$\mathrm{Hom}_{\mathrm{Vect}_k}(k^{(I)}, V)$

$= \mathrm{Hom}_{(\mathrm{Sets})}(I, V)$ for every k-v.s. V.

strictly speaking, we should write $G(V)$ here since we take all maps $I \to V$ of sets.

– In the previous example, we can replace k by any (commutative or non-commutative) ring R and Vect_k by the category of (left) R-modules, so that $R^{(I)}$ is the free R-module with basis e_i, $i \in I$.

Special case: $R = \mathbb{Z}$, free abelian groups.

- Let R be a ring. The forgetful functor $(R\text{-Alg}) \to (\text{Sets})$ has the following functor $(\text{Sets}) \to (R\text{-Alg})$ as a left adjt:

$$I \longmapsto R[X_i, i\in I] \quad \text{(polynomial ring)}$$

In fact, $\mathrm{Hom}_R(R[X_i, i\in I], A) = \mathrm{Hom}_{(\text{Sets})}(I, A)$ for every R-algebra A and set I.

- Let R be a ring, M an R-module. Then for all R-modules N, N' we have

$$\mathrm{Hom}(M\otimes_R N, N') = \mathrm{Hom}(N, \mathrm{Hom}(M,N')),$$

functorially in N, N', hence have an adjoint pair of functors

$$\begin{array}{ccc} (R\text{-Mod}) & \rightleftarrows & (R\text{-Mod}) \\ N & \longmapsto & M\otimes_R N \\ \mathrm{Hom}_R(M,N') & \longleftarrow & N' \end{array}$$

- Let $\varphi : R \to S$ be a ring homom.

$\rightsquigarrow$ $$\begin{array}{ccc} (R\text{-Mod}) & \rightleftarrows & (S\text{-Mod}) \\ M & \longmapsto & M \otimes_R S \\ \underbrace{N_{[\varphi]}}_{\substack{\text{view } N \text{ as an} \\ R\text{-module via } \varphi}} & \longleftarrow & N \end{array}$$ "base change"

and $\mathrm{Hom}_S(M \otimes_R S, N) = \mathrm{Hom}_R(M, N_{[\varphi]})$

for all M, N, functorially $\rightsquigarrow$ $- \otimes_R S \dashv -_{[\varphi]}$

adjoint pair

The notion of adjoint functors is useful since it is a concise description of a situation that occurs frequently when talking about functors. In addition, it has tangible consequences, as the following proposition shows.

Prop (1) Let $G: \mathcal{D} \to \mathcal{C}$ be a functor which has a left adjoint functor, i.e. G itself is right adjoint to some functor. Then G is <u>continuous</u>, i.e. commutes with limits in the following sense: For every partially ordered set I and projective system Y_i, $i \in I$, $\varphi_{ij}: Y_j \to Y_i$ $(j \leq i)$, the natural morphism

$$\lim_{i \in I} G(Y_i) \longrightarrow G(\lim_{i \in I} Y_i)$$

is an isomorphism (more precisely: whenever the limit $\lim_{i \in I} Y_i$ in $\mathcal{D}$ exists, then $G(\lim_{i \in I} Y_i)$ (together with the maps $G(\lim_{i \in I} Y_i) \to G(Y_i)$) is a limit of the system $G(Y_i)$, $G(\varphi)$).

(1') Let $\mathcal{C}, \mathcal{D}$ be abelian categories (e.g., categories of modules over some ring(s)), and let $G : \mathcal{D} \to \mathcal{C}$ be a functor which has a left adjoint. Then G is left exact.

(2) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which has a right adjoint. Then F is cocontinuous, i.e., commutes with colimits.

(2') Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between abelian categories which admits a left adjoint. Then F is right exact.

Proof
- (1') can be obtained as a 'formal consequence' of (1), similarly (2) $\Rightarrow$ (2').
- (1) and (2) are "dual" to each other, in particular either one of them follows formally from the other by "reversing all arrows", i.e., passing to the opposite categories

We omit the (easy) proof of (1) /(2) and only prove (1') directly for categories of modules ((2') can then be proved analogously).

So say $\mathcal{C} = (R\text{-Mod})$, $\mathcal{D} = (S\text{-Mod})$ for rings R, S, and let $G : \mathcal{D} \to \mathcal{C}$ be a functor which has a right adjoint F.

(we will assume G is additive, but one can show this is automatic since G has an adjoint)

Let $N' \to N \to N'' \to 0$ (*)

be a sequence of S-modules.

Recall the following fact from commutative algebra:

(*) exact $\iff$ for every S-module T the sequence $\mathrm{Hom}_S((*), T)$ is exact.

[Algebra 2] Satz 3.14, [Atiyah-Macdonald] Prop. 2.9

$\mathrm{Hom}_S(N'', T) \to \mathrm{Hom}_S(N, T) \to \mathrm{Hom}_S(N', T) \to 0$

To check that $G((*))$ $\left(:= G(N') \to G(N) \to G(N'') \to 0\right)$ is exact, we apply the same criterion for R-modules.

$\rightsquigarrow$ enough to show that

$\mathrm{Hom}(G((*)), T)$ exact for every R-mod. T

But $\mathrm{Hom}(G((*)), T) = \mathrm{Hom}((*), F(T))$
is exact by the above criterion, applied to the original sequence $(*)$ and the S-module $F(T)$.