IV Schemes

(reference: [GW] (2.9)–(2.12), Chapter 3.

IV.1 Locally ringed spaces

Have seen: to a ring A we can attach

- a topological space $\operatorname{Spec} A$, and
- a sheaf of rings $\mathcal{O}_A$

Would like to turn this into a functor

$$\operatorname{Spec} : (\text{Rings})^{op} \longrightarrow \left(\begin{array}{c}\text{"topol spaces with}\\ \text{sheaves of rings"}\end{array}\right)$$

such that Spec is fully faithful, i.e. induces bijections

$$\operatorname{Hom}_{(\text{Rings})}(A,B) \to \operatorname{Hom}_{LR}\big((\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}), (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})\big)$$

To achieve this, we need to

- define a suitable notion of morphism on the class of pairs (topol space, sheaf of rings)
- restrict to a suitable class of such pairs in order to ensure there are not too many morphisms on the right hand side

Def. A <u>ringed space</u> is a pair $(X, \mathcal{O}_X)$ consisting of a topol space X and a sheaf $\mathcal{O}_X$ of rings on X, called the <u>structure sheaf</u> of X.

A morphism between ringed spaces $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\flat)$ where f is a continuous map $X \to Y$ and $f^\flat$ is a morphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$ of sheaves of rings on Y.

<u>Example</u> X topol space, $X^{top} = (X, \mathcal{O}_X^{top})$ where $\mathcal{O}_X^{top}(U) = \{ U \to \mathbb{R} \text{ continuous} \}$ (with restriction of functions as restriction maps)

For $X \xrightarrow{f} Y$ a continuous map of topological spaces, obtain $\mathcal{O}_Y^{top} \xrightarrow{f^\flat} f_* \mathcal{O}_X^{top}$ by composition

$$\Big(\underset{f|_{f^{-1}(V)}}{f^{-1}(V) \longrightarrow V} \underset{s \in \mathcal{O}_Y^{top}(V)}{\longrightarrow \mathbb{R}} \Big) \in (f_* \mathcal{O}_X^{top})(V), \quad \underset{\text{open}}{V \subseteq Y}$$

$\leadsto$ morphism $X^{top} \to Y^{top}$ of ringed spaces.

Similarly: X differentiable / complex manifold,
$\mathcal{O}_X$ sheaf of differentiable fcts $U \to \mathbb{R}$ /
holomorphic functions $U \to \mathbb{C}$.

Given $f: X \to Y$ morphism of ringed spaces
(where we, as usual, do not mention $\mathcal{O}_X$, $\mathcal{O}_Y$, $f^\flat$
explicitly), from $f^\flat$ by adjunction we obtain

$$f^\#: f^{-1}\mathcal{O}_Y \to \mathcal{O}_X \quad \left(\begin{array}{c}\text{morph of}\\ \text{sheaves on } X\end{array}\right)$$

and therefore, for every $x \in X$ a ring homom

$$f^\#_x: \mathcal{O}_{Y,f(x)} = (f^{-1}\mathcal{O}_Y)_x \longrightarrow \mathcal{O}_{X,x}$$

(in the above examples, this is again composition
of "germs of functions" (i.e. fcts defined in an open
neighborhood of $f(x)$) with f).

In the above examples (when the structure sheaf is defined in terms of functions to a field k) we have a natural map $\mathcal{O}_{X,x} \xrightarrow{ev_x} k$ by evaluating a function at x, which is surjective and such that $\mathcal{O}_{X,x} \setminus \ker(ev_x) = \mathcal{O}_{X,x}^{\times}$.

Therefore the following is a natural definition:

Def. A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that for every $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a <u>local</u> ring, i.e., has a unique maximal ideal.

Given locally ringed spaces $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is a morphism $(f, f^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces such that for every $x \in X$ the ring homom. $f_x^{\#} : \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X,x}$ is a <u>local</u> ring homom., i.e. the image of the max'l ideal of $\mathcal{O}_{Y,f(x)}$ is contained in the max'l ideal of $\mathcal{O}_{X,x}$.

6.12.2022

Have defined

(Ringed Sp) category of ringed spaces

(Loc Ringed Sp) category of locally ringed spaces

<u>Remark</u> (local ring homomorphisms)

Let $\varphi: A \to B$ be a ring homomorphism between local rings A, B with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$.

Then φ local $:\iff \varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$

$\iff \varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$

$\iff \varphi$ fits into a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A/\mathfrak{m}_A \arrow[r] & B/\mathfrak{m}_B \end{tikzcd},$$

i.e. φ induces a homomorphism $\kappa(\mathfrak{m}_A) \longrightarrow \kappa(\mathfrak{m}_B)$ between the residue class fields.

<u>Example</u> Let k be a field, and let

(k-LocRingedSp) be the category of locally ringed spaces $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is (not just a sheaf of rings, but) a sheaf of k-algebras, and for morphisms $(f, f^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ we require that $f^\flat$ is a morphism of sheaves of k-algebras.

Let $\mathcal{C}$ be one of the following categories

(Top) cat. of topological spaces, k any topological field,
(DiffMf) cat. of differentiable manifolds, $k = \mathbb{R}$,
(CplMf) cat. of complex manifolds, $k = \mathbb{C}$,

As in the previous example we obtain a functor $\mathcal{C} \to$ (k-LocRinged Sp)

(where for $X \in \mathrm{Ob}(\mathcal{C})$, $U \subseteq X$ open,

$$\mathcal{O}_X(U) = \begin{cases} \{\text{continuous fcts } U \to k\} \\ \{\text{differentiable functions } U \to \mathbb{R}\} \\ \{\text{holomorphic functions } U \to \mathbb{C}\} \end{cases}).$$

In all three cases, this functor is fully faithful.

(For details see [Wedhorn, Manifolds, Sheaves, and Cohomology], Example 4.5.)

By what we have done so far, we can attach to every affine scheme a locally ringed space.

<u>Proposition</u> Mapping a ring A to the pair $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ and a ring homomorphism $A \xrightarrow{\varphi} B$ to the pair $(\varphi^a, \varphi^b : \mathcal{O}_{\operatorname{Spec} A} \to (\varphi^a)_* \mathcal{O}_{\operatorname{Spec} B})$ where

on $D(s) \subseteq \operatorname{Spec} A$, $s \in A$, φ^b is given by

$$A_s \to B_{\varphi(s)}, \quad \frac{a}{s^i} \mapsto \frac{\varphi(a)}{\varphi(s)^i},$$

note that $(\varphi_a)^{-1}(D(s)) = D(\varphi(s))$

defines a functor

$$(\text{Rings})^{op} \to (\text{LocRingedSp})$$

Will see below: this functor is fully faithful.

<u>Proof</u> • for a ring A and $\mathfrak{p} \in \operatorname{Spec} A$, the stalk of $\mathcal{O}_{\operatorname{Spec} A}$ at $\mathfrak{p}$ is $A_{\mathfrak{p}}$ which is a local ring,

• For a ring homom. $\varphi : A \to B$ and $\mathfrak{q} \in \operatorname{Spec} B$, the attached morphism

$$(f, f^{\flat}) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$$

(which clearly is a morphism of ringed spaces) gives rise to a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \arrow[r] \arrow[d] & \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B) \arrow[d] \\
\mathcal{O}_{\operatorname{Spec} A, f(\mathfrak{q})} \arrow[r, "f^{\#}_{\mathfrak{q}}"] & \mathcal{O}_{\operatorname{Spec} B, \mathfrak{q}}
\end{tikzcd}$$

which we can rewrite as

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\
A_{\varphi^{-1}(\mathfrak{q})} \arrow[r, "f^{*}_{\mathfrak{q}}"] & B_{\mathfrak{q}}
\end{tikzcd}$$

It is then clear that $f^{\#}_{\mathfrak{q}}(\varphi^{-1}(\mathfrak{q}) A_{\varphi^{-1}(\mathfrak{q})}) \subseteq \mathfrak{q} B_{\mathfrak{q}}$ because $\varphi(\varphi^{-1}(\mathfrak{q})) \subseteq \mathfrak{q}$.

Hence the morphism we have defined is a morphism of <u>locally</u> ringed spaces.

IV.2 Schemes

Def An *affine scheme* is a locally ringed space that is isomorphic to a locally ringed space of the form $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A. A morphism of affine schemes is a morphism of locally ringed spaces.

Def A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ s.t. there exists an open cover $X = \bigcup_{i \in I} U_i$ such that for every $i \in I$, the locally ringed space $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme.

A morphism of schemes is a morphism of locally ringed spaces.

(Ancient terminology: prescheme)

Notation: (Aff) category of affine schemes

(Sch) category of schemes

We obtain

7.12.2022

$$(\mathrm{Rings})^{op} \xrightarrow{\sim} (\mathrm{Aff}) \longrightarrow (\mathrm{Sch}) \longrightarrow (\mathrm{Loc\ Ringed\ Sp})$$

equivalence of categories

"full subcategory", i.e. same Hom-sets

Proposition The functor $\mathrm{Spec}: (\mathrm{Rings})^{op} \to (\mathrm{Aff})$

$$A \longmapsto (\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A})$$

is an equivalence of categories.

Proof. We will show that the functor

$\Gamma : (\mathrm{Aff}) \to (\mathrm{Rings})^{op}$, given

on objects: $(X, \mathcal{O}_X) \longmapsto \Gamma(X, \mathcal{O}_X) := \mathcal{O}_X(X)$

"ring of global sections"

on morphisms: $\left((f, f^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\right) \longmapsto \Gamma(Y, \mathcal{O}_Y) \xrightarrow{f^\flat(Y)} \Gamma(Y, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$

is a quasi-inverse of Spec. Clearly $\Gamma \circ \mathrm{Spec} = \mathrm{id}$.

By definition of the category of affine schemes, every affine scheme is of the form $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ up to isomorphism, and clearly

$$\operatorname{Spec}\left(\Gamma\left((\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})\right)\right) \cong \operatorname{Spec} A.$$

It is therefore enough to show that the functors Spec and Γ induce bijections (inverse to each other)

$$\operatorname{Hom}_{\mathrm{Ring}}(A,B) \rightleftarrows \operatorname{Hom}_{\mathrm{locRingdSp}}(\operatorname{Spec} B, \operatorname{Spec} A)$$

(where we write $\operatorname{Spec} A$ for the loc. ringed space $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, similarly for $\operatorname{Spec} B$).

To this end, it remains to show that $\operatorname{Spec}(\Gamma(f)) = f$ for every morphism $f = (f, f^\flat) : \operatorname{Spec} B \to \operatorname{Spec} A$ of loc. ringed spaces. Write $\varphi = \Gamma(f) : A \to B$.

For every $x \in \operatorname{Spec} B$, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A_{f(x)} \arrow[r, "f_x^\#"] & B_x \end{tikzcd} \qquad (*)$$

Since $f_x^\#$ is local, it follows that $\underbrace{f(x) = \varphi^{-1}(x)}_{\substack{\text{view } x \subseteq B \text{ as} \\ \text{a prime ideal in } B}}$

$\Rightarrow$ $f = \varphi^a$ (as continuous maps).

To check the equality of the two sheaf morphisms $\mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$ $(= \varphi^a_* \mathcal{O}_{\operatorname{Spec} B})$, by adjunction we may pass to $f^{-1}\mathcal{O}_{\operatorname{Spec} A} \to \mathcal{O}_{\operatorname{Spec} B}$.

Further, it is enough to check that both sheaf morphisms induce the same homomorphism on all stalks at points $x \in B$. But this follows from the commutative diagram $(*)$.

Open subschemes

Lemma Let X be a scheme, and let U be an open subset of the topological space X. Then $(U, \mathcal{O}_{X|U})$ is a scheme.

Schemes of this form are called <u>open subschemes</u> of X.

Proof Let $x \in U$, $V \subseteq X$ an open neighborhood of x s.t. $(V, \mathcal{O}_{X|V})$ is an affine scheme ($\cong \operatorname{Spec} \mathcal{O}_X(V)$).

Then there exists $s \in \mathcal{O}_X(V)$ s.t. $x \in D(s) \subseteq V \cap U$

$\rightsquigarrow$ $D(s)$ is an open neighborhood of x in U

and $(D(s), \mathcal{O}_{U|D(s)}) = (D(s), \mathcal{O}_{X|D(s)})$

is an affine scheme.

So U has an open cover by affine schemes.

If $U \subseteq X$ is an open subscheme (where as usual we omit the structure sheaves from the notation), then there is a natural scheme morphism $U \xrightarrow{\iota} X$ $\left(\begin{array}{l}\text{inclusion } \iota \text{ on topol sp.,} \\ \text{res}: \mathcal{O}_X(V) \to \mathcal{O}_X(V \cap U) \text{ on sheaves}\end{array}\right)$

13.12.2002

Def. A scheme morphism $(f, f^\flat) : Y \to X$ is called an <u>open immersion</u> if f is a homeomorphism from Y onto an open subset $U \subseteq X$ — so that f factors as $Y \xrightarrow[\sim]{\dot{f}} U \xhookrightarrow{\iota} X$ — and $f^\flat$ induces an isomorphism $\mathcal{O}_{X|U} \xrightarrow{\sim} \dot{f}_* \mathcal{O}_Y$.

Examples

- R ring $\rightsquigarrow$ $\mathbb{A}^n_R := \operatorname{Spec} R[T_1, \dots, T_n]$

  "affine n-space over R" or

  "affine space of relative dimension n over R"

  $\rightsquigarrow$ $\mathbb{A}^n_R \longrightarrow \operatorname{Spec} R$ morphism of affine schemes

- k a field $\rightsquigarrow$ $0 := (T_1, \dots, T_n) \subset k[T_1, \dots, T_n]$ "origin of $\mathbb{A}^n_k$"

  closed point

  $\rightsquigarrow$ $\mathbb{A}^n_k \setminus \{0\} \subseteq \mathbb{A}^n_k$ open subscheme.

  Can show: for $n \geq 2$, $\mathbb{A}^n_k \setminus \{0\}$ is a scheme which is not affine.

- A a domain, $X = \operatorname{Spec} A \ni \eta$ the generic point

  ($\leftrightarrow$ the zero ideal $\subset A$)

  Then $\mathcal{O}_{X,\eta} = \operatorname{Frac}(A) =: K$ and for every non-empty open $U \subseteq X$ the natural map $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,\eta}$ is injective and identifies $\mathcal{O}_X(U)$ with a subring of K. We have $\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x}$ inside K.

- A a ring, $s \in S$ $\rightsquigarrow$ $\tau : A \to A_s$, $a \mapsto \frac{a}{1}$

Obtain $\operatorname{Spec} A_s \to \operatorname{Spec} A$ morphism of affine schemes.

This morphism factors as

$$\begin{tikzcd} \operatorname{Spec} A_s \arrow[rr] \arrow[dr, "\cong"'] & & \operatorname{Spec} A \\ & (D(s), \mathcal{O}_{\operatorname{Spec} A}|_{D(s)}) \arrow[ur] & \end{tikzcd}$$

open immersion

- A a ring, $\mathfrak{a} \subseteq A$ an ideal $\rightsquigarrow$ $\pi : A \to A/\mathfrak{a}$
canonical projection

Obtain $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$ morphism of affine schemes which on topological spaces factors as

$$\begin{tikzcd} \operatorname{Spec} A/\mathfrak{a} \arrow[rr] \arrow[dr, "g"', "\cong"] & & A \\ & V(\mathfrak{a}) \arrow[ur] & \end{tikzcd}$$

$\rightsquigarrow$ $V(\mathfrak{a}) := (V(\mathfrak{a}), g_* \mathcal{O}_{\operatorname{Spec} A/\mathfrak{a}})$ scheme

("closed subscheme of $\operatorname{Spec} A$" — we will later introduce a general notion of closed subscheme of an arbitrary scheme)

- k a field, $n \geq 0$, $A_n = \operatorname{Spec} k[T]/(T^{n+1})$

$\rightsquigarrow$ for all n, $\operatorname{Spec} A_n$ has a single point

We have scheme morphisms

$$\operatorname{Spec} k = \operatorname{Spec} A_0 \longrightarrow \underbrace{\operatorname{Spec} A_1 \to \dots \to \operatorname{Spec} A_n \to \dots}\to \mathbb{A}^1_k$$

- on topological spaces, all these maps are identity
- none of these maps is an isomorphism of schemes

If we view elements $f \in k[T]$ as "functions" on $\mathbb{A}^1_k$, then the ring $k[T]/(T^{n+1})$ "remembers" the value $f(0)$ of the function and all derivatives $f'(0), f''(0) \dots$ up to order n. Heuristically (viewing derivatives as limits in the sense of analysis) we need a "small infinitesimal neighborhood" to compute these derivatives. In algebraic geometry we cannot 'see' these infinitesimal neighborhoods on the level of topological spaces, but they are encoded in some sense by the structure sheaf.

- For A a ring, $\mathfrak{a}, \mathfrak{b} \subseteq A$ ideals, we define the scheme-theoretic intersection of $V(\mathfrak{a})$, $V(\mathfrak{b})$ as $V(\mathfrak{a}) \cap V(\mathfrak{b}) := \operatorname{Spec} A/_{\mathfrak{a}+\mathfrak{b}}$.

On topological spaces this is just the usual intersection, but the above definition also provides us with a structure sheaf and hence with finer information.

For example, for k a field, $A = k[X,Y]$,

- $\mathfrak{a} = (Y)$, $\mathfrak{b} = (X-Y)$

  $\rightsquigarrow V(\mathfrak{a}) \cap V(\mathfrak{b}) \cong \operatorname{Spec} k$

- $\mathfrak{a} = (Y)$, $\mathfrak{b} = (X^2 - Y)$

  $\rightsquigarrow V(\mathfrak{a}) \cap V(\mathfrak{b}) \cong \operatorname{Spec} k[X]/_{(X^2)}$

IV.3 Scheme morphisms

14.12.2022

<u>Notation</u> X topol space, $\mathcal{F}$ a (pre-)sheaf on X.

One often writes $\Gamma(U, \mathcal{F})$ for the set $\mathcal{F}(U)$ of $\mathcal{F}$ on an open subset $U \subseteq X$.

An important justification why the notion of locally ringed space works well (and better than 'ringed space') for us, is the following result which generalizes the statement that $(\text{Rings})^{op} \simeq (\text{Aff})$ is an equivalence.

<u>Theorem</u> Let A be a ring, and let $(X, \mathcal{O}_X)$ be a locally ringed space. Then the map

$$\mathrm{Hom}_{\text{LocRingedSp}}\left((X,\mathcal{O}_X), (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})\right) \xrightarrow{\Gamma} \mathrm{Hom}_{\text{Ring}}(A, \Gamma(X,\mathcal{O}_X))$$

is a bijection.

<u>Proof</u> It is possible to prove the theorem along the same lines as the proof above of the special case when X is an affine scheme, as well.

Here we will give a different proof under the assumption that X is a scheme. Namely, the following proposition will allow us to reduce to the previous case (i.e. the case that X is an affine scheme).

<u>Prop</u> (Gluing of morphisms)

(1) Let X be a set, $U_i \subseteq X$ subsets, $i \in I$, s.t. $X = \bigcup U_i$.

If Y is any set and $(f_i : U_i \to Y)_{i \in I}$ is a family of maps s.t. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a unique map $f: X \to Y$ s.t. $f|_{U_i} = f_i$ for all i.

(2) The same statement holds for topological spaces X, Y, open covers $X = \bigcup_{i \in I} U_i$, continuous maps $f_i : U_i \to Y$ and $f : X \to Y$

(3) The same statement holds for ringed spaces, open covers and morphisms of ringed spaces.

(4) The same statement holds for locally ringed spaces, open covers, and morphisms of locally ringed spaces.

(In (3), (4) more precisely the U_i are open subsets of the topological space X and are viewed as (locally) ringed spaces by equipping them with the structure sheaf $\mathcal{O}_{U_i} := \mathcal{O}_{X|U_i}$.

For a morphism $f: X \to Y$ of (locally) ringed spaces and an open $U \subseteq X$, the restriction $f|_U$ is defined as

- the usual restriction on topological spaces
- on sheaves, $\mathcal{O}_Y \to (f|_U)_* \mathcal{O}_U$ given by

$$
\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "f^\flat(V)"'] & \mathcal{O}_X(f^{-1}(V)) \arrow[r, "\mathrm{res}"] & \mathcal{O}_X(f^{-1}(V) \cap U) \arrow[d, equal] \\
& & \mathcal{O}_U(f|_U^{-1}(V)) \arrow[d, equal] \\
& & ((f|_U)_* \mathcal{O}_U)(V).
\end{tikzcd}
$$

Remark.

Equivalently, we can express the proposition by saying that the presheaf $U \mapsto \mathrm{Hom}_{\mathcal{C}}(U, Y)$ on X is a sheaf (for every fixed $Y \in \mathrm{Ob}\,\mathcal{C}$), when $\mathcal{C}$ is the category of sets / topological sp. / ringed sp. / locally ringed spaces (and in the first case we equip every set with the discrete topology in order to talk of sheaves on a set X).

Proof The proof is easy and we only give a few hints.

- For $x \in X$ define $f(x) := f_i(x)$ for any i with $x \in U_i$. By the assumption, the value is independent of the choice of i.

- If all f_i are continuous, then it follows (since $X = \bigcup U_i$ and $f|_{U_i} = f_i \ \forall i$) that f is continuous (Cf. Problem 25, problem sheet 7)

- If X, Y are (locally) ringed spaces, define $f^\flat : \mathcal{O}_Y \to f_* \mathcal{O}_X$ as follows: for $V \subseteq Y$ open, we have

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathcal{O}_X(f^{-1}(V)) \arrow[r] & \prod_i \mathcal{O}_X(f^{-1}(V) \cap U_i) \arrow[r] & \prod_{i,j} \mathcal{O}_X(f^{-1}(V) \cap U_i \cap U_j) \\
& & \mathcal{O}_Y(V) \arrow[u] \arrow[ur, "=0"'] &
\end{tikzcd}$$

exact since $\mathcal{O}_X$ is a sheaf

from the $f_i^\flat$

$= 0$ since $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ as morphisms of (loc.) ringed spaces

It follows that the map $\mathcal{O}_Y(V) \to \prod_i \mathcal{O}_X(f^{-1}(V) \cap U_i)$ factors through $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$. One checks that this gives the desired map $\mathcal{O}_Y \to f_* \mathcal{O}_X$.

It is clear that the morphism f constructed in this way is a morphism of locally ringed spaces iff X, Y are locally ringed spaces and all f_i are morphisms of locally ringed spaces.

<u>Proof of theorem</u> $\quad \mathrm{Hom}(X, \mathrm{Spec}\,A) \xrightarrow{\sim} \mathrm{Hom}(A, \Gamma(X, \mathcal{O}_X))$

Let $X = \bigcup_{i \in I} U_i$ be an open cover by affine schemes. We know already that for every affine scheme U, $\mathrm{Hom}(U, \mathrm{Spec}\,A) \xrightarrow{\sim} \mathrm{Hom}(A, \Gamma(U, \mathcal{O}_X))$ $\circledast$

For $i, j \in I$ let $U_i \cap U_j = \bigcup_{i,j,k} U_{ijk}$ be an open cover by affine schemes.

Observe that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(X, \mathrm{Spec}\,A) \arrow[r] \arrow[d] & \prod_i \mathrm{Hom}(U_i, \mathrm{Spec}\,A) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal, "\circledast"] & \prod_{i,j,k} \mathrm{Hom}(U_{ijk}, \mathrm{Spec}\,A) \arrow[d, equal, "\circledast"] \\
\mathrm{Hom}(A, \Gamma(X, \mathcal{O}_X)) \arrow[r] & \prod_i \mathrm{Hom}(A, \Gamma(U_i, \mathcal{O}_X)) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j,k} \mathrm{Hom}(A, \Gamma(U_{ijk}, \mathcal{O}_X))
\end{tikzcd}$$

By the proposition on gluing, the upper row of this diagram are exact. By the sheaf property of $\mathcal{O}_X$ and since the functor $\mathrm{Hom}(A, -)$ is left exact, the lower row is exact. So the left vertical map is an isom.

Notation We usually write X as shorthand notation for $(X, \mathcal{O}_X) \in \mathrm{Ob}\, \mathrm{LocRingedSp}$, and in particular consider $\mathrm{Spec}\, A$ (for A a ring) as the locally ringed space $(\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A})$.

Example • Every scheme X admits a unique morphism $X \to \mathrm{Spec}\, \mathbb{Z}$

• If k is a field, then

$$X \to \mathrm{Spec}\, k \quad \hat{=} \quad k\text{-algebra structure on } \Gamma(X, \mathcal{O}_X)$$

Once a morphism $X \to \mathrm{Spec}\, k$ is fixed, all $\Gamma(U, \mathcal{O}_X)$, $\mathcal{O}_{X,x}$, $\kappa(x)$ are equipped with a k-algebra structure in a natural way.

Given schemes X, Y with morphisms $X \to \mathrm{Spec}\, k$, $Y \to \mathrm{Spec}\, k$ it is usually reasonable to restrict attention to those morphisms which are compatible with these k-algebra structures.

This leads to the following definition.

Relative schemes / schemes over a base scheme

Let S be a scheme.

We define the category (Sch/S) of S-schemes as follows: **objects** scheme morphisms $X \to S$

morphisms morphisms from $X \xrightarrow{f} S$ to $Y \xrightarrow{g} S$ are scheme morphisms $X \xrightarrow{h} Y$

s.t.
$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & f \searrow \quad \swarrow g & \\ & S & \end{array}$$
commutes

(with the obvious composition of morphisms).

Terminology: • S is often called the "base scheme"

• Objects $X \to S$ of (Sch/S) are called S-schemes or schemes over S, and the morphism to S (which is often not written explicitly) is called the structure morphism

If $S = \operatorname{Spec} R$, we usually write (Sch/R) for (Sch/S) and speak of R-schemes (or of schemes over R).

Example $(Sch/\mathbb{Z}) = (Sch)$.

For a ring R and R-schemes X, Y and a morphism $X \xrightarrow{f} Y$ of R-schemes, all ring homomorphisms attached to f "in a natural way" are R-algebra homomorphisms

(e.g. $f^\flat(Y) : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$,

$f^\sharp_x : \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X,x}$,

$\kappa(f(x)) \to \kappa(x)$ for $x \in X$)

Def (T-valued points of a scheme)

(1) Let X be a scheme. For every scheme T we write $X(T) := \mathrm{Hom}_{Sch}(T, X)$ for the set of scheme morphisms from T to X and call this the <u>set of T-valued points of X</u>

(2) If X, T in (1) are S-schemes for some scheme S, i.e., are equipped with structure morphisms $X \to S$, $T \to S$, we usually (by abuse of notation) write

$$X(T) := \mathrm{Hom}_{(Sch/S)}(T, X)$$

(and again call this the set of T-valued points of the S-scheme X).

If in the above definition $T = \mathrm{Spec}\, R$ is affine, we also write $X(R)$ for $X(T)$ and speak of <u>R-valued points</u>

Example R a ring, consider $\operatorname{Spec} R$, $\mathbb{A}^n_R = \operatorname{Spec} R[T_1, \dots, T_n]$ as R-schemes in the obvious way. Then

$$\mathbb{A}^n_R(R) = \operatorname{Hom}_{(Sch/R)}(\operatorname{Spec} R, \mathbb{A}^n_R)$$

$$= \operatorname{Hom}_{R\text{-Alg}}(R[T_1, \dots, T_n], R) \cong R^n$$

$$\varphi \longmapsto (\varphi(T_i))_i$$

and more generally for every R-scheme T,

$$\mathbb{A}^n_R(T) = \operatorname{Hom}_{R\text{-Alg}}(R[T_1, \dots, T_n], \Gamma(T, \mathcal{O}_T)) = \Gamma(T, \mathcal{O}_T)^n.$$

20.12.2022

Recall • X a scheme, T a scheme,

$$X(T) := \mathrm{Hom}_{Sch}(T, X) \quad \text{"T-valued points of X"}$$

• S a scheme, X, T schemes $/S$,

$$X(T) := \mathrm{Hom}_{Sch/S}(T, X) \quad \text{"T-valued pts of X"}$$

special case $T = \mathrm{Spec}\, R$: $X(R) := X(\mathrm{Spec}\, R)$, R-valued points

Remark (1) R a ring, $f_1, \dots, f_r \in R[T_1, \dots, T_n]$,

$$X := V(f_1, \dots, f_r) := \mathrm{Spec}\, R[T_1, \dots, T_n]/(f_1, \dots, f_r)$$

$$\rightsquigarrow \begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{A}^n_R \arrow[d] \\ & \mathrm{Spec}\, R \end{tikzcd}$$

While the scheme and even the topological space X are kind of hard to describe, the sets of A-valued pts of X (for R-algebras A) are just the sets of solutions of $f_1 = \dots = f_r = 0$: $X(A) = \{(t_i)_i \in A^n;\ f_j(t_1, \dots, t_n) = 0\ \forall j\}$

$$\begin{array}{ccc} X(A) & = & \{(t_i)_i \in A^n;\ f_j(t_1,\dots,t_n)=0\ \forall j\} \\ \cap & & \cap \\ \mathbb{A}^n_R(A) & = & A^n \end{array}$$

(2) More precisely, for every scheme X, taking T-valued points for varying T gives us a functor $h_X : (\mathrm{Sch})^{op} \to (\mathrm{Sets})$

$$\begin{aligned} T &\longmapsto X(T) \\ T' \xrightarrow{g} T &\longmapsto X(T) \to X(T') \\ & \qquad\quad f \mapsto f \circ g \end{aligned}$$

It is not hard to show (and we will come back to this later) that the functor h_X determines the scheme X (in the sense that given schemes X, Y, there exists an isomorphism $h_X \cong h_Y$ if and only if $X \cong Y$ as schemes).

Morphisms from the spectrum of a field into a scheme

(I) Let X be a scheme, $x \in X$, $U \subseteq X$ an affine open neighborhood of x, say $U = \operatorname{Spec} A$, $\mathfrak{p} \subset A$ the prime ideal corresp. to x.

We obtain scheme morphisms

$$j_x : \operatorname{Spec} \mathcal{O}_{X,x} = \operatorname{Spec} A_{\mathfrak{p}} \to \operatorname{Spec} A = U \hookrightarrow X,$$

$$i_x : \operatorname{Spec} \kappa(x) \to \operatorname{Spec} \mathcal{O}_{X,x} \xrightarrow{j_x} X$$

which are independent of the choice of U.

The image of i_x (on the level of topological spaces) is $\{x\}$.

The map j_x on topological spaces induces a homeomorphism $\operatorname{Spec} \mathcal{O}_{X,x} \xrightarrow{\sim} \{x' \in X;\ x \in \overline{\{x'\}}\}$.

(II) Let X be a scheme, K a field.

Then $\operatorname{Spec} K$ as a topological space consists of a single point.

Let $\operatorname{Spec} K \xrightarrow{f} X$ be a morphism of schemes with image $\{x\}$. Then f induces a map $\kappa(x) \to K$ on residue class fields and therefore factors through i_x.

We obtain bijections (for fixed K):

- $$\begin{aligned} \operatorname{Hom}(\kappa(x), K) &\xrightarrow{1:1} \{ f : \operatorname{Spec} K \to X ;\ \operatorname{im}(f) = \{x\} \} \\ \varphi &\longmapsto i_x \circ \operatorname{Spec}(\varphi) \end{aligned}$$

- $$\{ (x, \varphi) ;\ x \in X,\ \underset{\text{ring homom}}{\varphi : \kappa(x) \to K} \} \xrightarrow{1:1} \operatorname{Hom}(\operatorname{Spec} K, X) = X(K)$$

Similarly if k is a field, X a k-scheme, K/k a field extension (i.e., $\operatorname{Spec} K$ is a $\operatorname{Spec} k$-scheme):

$$\{ (x, \varphi) ;\ x \in X,\ \underset{k\text{-homom}}{\varphi : \kappa(x) \to K} \} \xrightarrow{1:1} X(K)$$ (in the sense of k-schemes)