VI Further properties of schemes

VI.1 Topological properties, noetherian schemes

Def (1) A scheme is called

connected / quasi-compact / irreducible

if its underlying topological space has this property.

(2) A morphism of schemes is called

injective / surjective / bijective /

open* / closed** / a homeomorphism,

if the corresponding continuous map has this property.

* $f: X \to Y$ open: $\forall U \subseteq X$ open: $f(U) \subseteq Y$ open

** $f: X \to Y$ closed: $\forall Z \subseteq X$ closed: $f(Z) \subseteq Y$ closed

Def A morphism $f: X \to Y$ of schemes is called

quasi-compact, if for every quasi-compact open subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.

Noetherian schemes

Def (1) A topological space is called noetherian if it satisfies the descending chain condition for closed subsets.

(2) A scheme X is called locally noetherian if it admits an affine open cover $X = \bigcup_{i \in I} U_i$ s.t., for every i, $\Gamma(U_i, \mathcal{O}_X)$ is a noetherian ring.

(3) A scheme X is called noetherian if it is locally noetherian and quasi-compact.

Remark (0) X topol space. Then

X noeth $\Leftrightarrow$ every non-empty set of closed subsets of X has a minimal elt $\Leftrightarrow$ every non-empty set of open subsets of X has a maximal element

(1) X noeth topol space. Then every subspace of X is noetherian.

(2) X topol space. Then

X noetherian $\iff$ every open subset of X is quasi-compact

(3) X noetherian topol space

$\Rightarrow$ X has only finitely many irreducible components.

(4) X noeth scheme $\Rightarrow$ the underlying topol space of X is noetherian

(but $\Leftarrow$ does not hold in general!)

(For proofs, see [GW, Section (1.7)], for instance.)

Prop Let $X = \operatorname{Spec} A$ be an affine scheme.

Then X noetherian scheme $\iff$ A noetherian ring

Proof "$\Leftarrow$" is clear by definition.

"$\Rightarrow$". Let $X = \bigcup U_i$ an affine open cover with all U_i noetherian. Fix an index i. For $f \in A$ with $D(f) \subseteq U_i$, we have $D(f) = D_{U_i}(f|_{U_i}) = \operatorname{Spec}(\Gamma(U_i, \mathcal{O}_X)_{f|_{U_i}})$. Since localizations of noeth. rings are noeth., it follows that $D(f)$ is noetherian.

Replacing the U_i's by (several) $D(f_j)$'s we may therefore assume that each U_i is a principal open in X, say $U_i = D(f_i)$.

The following lemma then implies that every ideal $I \subseteq A$ is finitely generated, and hence that A is noetherian.

Lemma Let A be a ring, $f_1, \dots, f_r \in A$ s.t. $(f_1, \dots, f_r) = A$ $\left(\iff \bigcup_{i=1}^{r} D(f_i) = \operatorname{Spec} A\right)$.

Let M be an A-module such that for every $i = 1, \dots, r$ the localization M_{f_i} is a finitely generated A_{f_i}-module.

Then M is a finitely generated A-module.

Proof We write $M_{f_i} = \left\langle \frac{m_{ij}}{f_i^{n_j}}, j = 1, \dots, r_i \right\rangle_{A_{f_i}}$

and let $N := \langle m_{ij}, i, j \rangle \subseteq M$.

Claim $N = M$ (hence M fin. gen., as desired)

In fact, enough to show that $N_p = M_p$ for every prime ideal $p \in \operatorname{Spec} A$ (then $(M/N)_p = M_p / N_p = 0 \ \forall p$ which implies $M/N = 0$).

But for $p \in \operatorname{Spec} A$, say $p \in D(f_i)$, we have $N_p = (N_{f_i})_p = (M_{f_i})_p = M_p$.

Prop Let X be a (locally) noetherian scheme and let $U \subseteq X$ be an open subscheme. Then U is (locally) noetherian.

Proof
- Clear for "locally noeth" since localizations of noetherian rings are noetherian, so principal open subschemes of Spec of a noeth. ring are noetherian.
- For "noetherian" use that every open subspace of a noetherian topological space is quasi-compact.

Cor X a locally noetherian scheme, $U \subseteq X$ affine open subscheme $\Rightarrow$ $\Gamma(U, \mathcal{O}_X)$ is a noetherian ring.

Example (A noetherian scheme X s.t. $\Gamma(X, \mathcal{O}_X)$ is not a noetherian ring)

[M. Ojanguren, Un ouvert bizarre]

This was not discussed in the lectures

Let k be a field,

$A, B \subseteq \mathbb{P}^3_k$ projective planes which intersect in a line L

(e.g. $A = V_+(X)$, $B = V_+(Y)$ where X, Y, Z, W are homog. coord. on $\mathbb{P}^3_k$)

Let $D \neq L$ be a projective line $\subseteq A$ which intersects L in a point P (e.g. $D = V_+(X, Z)$, $P = (0:0:0:1)$)

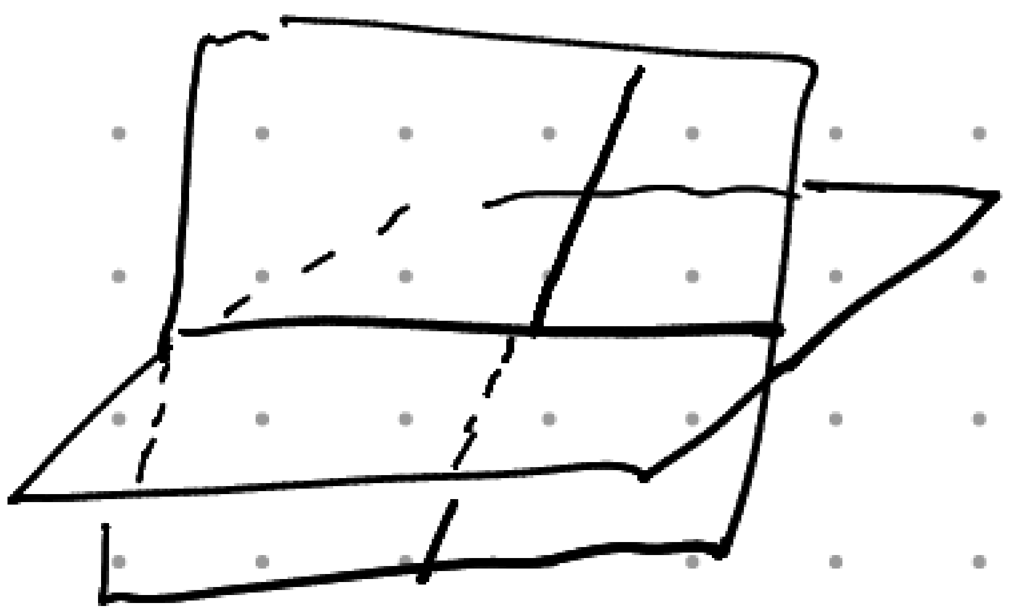
Let $U = (A \cup B) \setminus D$, a noetherian scheme.

Then $\Gamma(U, \mathcal{O}_X) \underset{①}{\cong} \{ f \in k[x,y] ;\ f(x,0) = f(0,0) \}$

is $\underbrace{\text{not noetherian}}_{②}$

① Cover $U = (A \cup B) \setminus D = (A \setminus D) \cup (B \setminus \{P\})$

($\subset A \cup B$ open) $= (A \cap U) \cup (B \cap U)$



$$\left.\begin{array}{ll} \Gamma(B \setminus \{P\}) = \Gamma(B) = k & \\ \Gamma(A \setminus D) = k[x,y] & y \\ \quad \downarrow & \downarrow \\ \Gamma(L \setminus \{P\}) = k[x] & 0 \end{array}\right\} \Longrightarrow ①$$

[...]

② $R = \{ f \in k[x,y] \, ; \, f(x,0) \text{ constant} \} =$

$\{ f \in k[x,y] \, ; \, f(x,0) = f(0,0) \} = k + \underbrace{y \cdot k[x,y]}_{\text{maximal ideal, not finitely generated}}$

$\boxed{17.1.23}$

Generic points

Prop Let X be a scheme, $Z \subseteq X$ an irreducible closed subset of X.

Then there exists a unique point $\eta_Z \in Z$ s.t. $\overline{\{\eta_Z\}} = Z$ (called the <u>generic point</u> of Z).

Proof We have already proved this for X affine.

In the general case, let $U \subset X$ affine open s.t. $Z \cap U = \emptyset$,

then $Z \cap U \subseteq Z$ open, hence dense, and

$Z \cap U \subseteq U$ is closed and irreducible (since its closure in X is $= Z$, so irreducible)

Let $\eta_Z \in Z \cap U$ be its (unique) generic point.

Then $\overline{\{\eta_Z\}}^X = Z$. Uniqueness follows similarly since a generic pt lies in every non-empty open.

Reduced and integral schemes

Recall R a ring,

$$\mathrm{nil}(R) = \{ f \in R;\ \exists n \geq 0 : f^n = 0 \} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} R} \mathfrak{p}.$$

R reduced $:\Longleftrightarrow \mathrm{nil}(R) = 0$

$\Longleftrightarrow$ the natural map $R \to \prod_{\mathfrak{p} \in \operatorname{Spec} R} \kappa(\mathfrak{p})$ is injective

("elem. of R, seen as functions on $\operatorname{Spec} R$, are determined by their values")

Def A scheme X is called reduced if the following equivalent conditions are satisfied:

(i) For every open $U \subseteq X$ the ring $\Gamma(U, \mathcal{O}_X)$ is reduced.

(ii) There exists an affine open cover $X = \bigcup_{i \in I} U_i$ s.t. for every $i \in I$ the ring $\Gamma(U_i, \mathcal{O}_X)$ is reduced.

(iii) For every $x \in X$ the stalk $\mathcal{O}_{X,x}$ is reduced.

(Proof of equivalence: Problem sheet 10, Pblm 39)

As a consequence of the above remark, we have:

<u>Cor</u> Let X be a reduced scheme. Then the natural map $\Gamma(X, \mathcal{O}_X) \xrightarrow{ev} \prod_{x \in X} \kappa(x)$ is injective.

<u>Proof</u> Let $X = \bigcup U_i$ be an affine open cover.

For $f \in \Gamma(X, \mathcal{O}_X)$ with $ev(f) = 0$, to show $f = 0$ it is enough to show $f|_{U_i} = 0$ for all i.

But the 'evaluation maps' ev for X and ev_i for U_i are compatible, and U_i is affine, so we reduce to the corresponding property of a reduced ring.

<u>Remark</u> There exist non-reduced schemes X such that $\Gamma(X, \mathcal{O}_X)$ is a reduced ring.

(Example: $X = V_+(X^2) \subseteq \mathbb{P}^2_k$, k a field.)

Def A scheme is called <u>integral</u>, if it is irreducible and reduced.

This notion of "integral" is closely related to the notion of <u>(integral) domain</u> (an affine scheme Spec A is integral if and only if A is a domain), but not to the notion of 'integral ring homomorphism.

<u>Proposition</u> Let X be a scheme. The following are equivalent:

(i) X is integral

(ii) for every non-empty open $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ is a domain

In this case all local rings $\mathcal{O}_{X,x}$ are domains.

<u>Proof</u> <u>(i) ⇒ (ii)</u>. Every non-empty open subscheme of X is itself integral, so it is enough to prove that $\Gamma(X, \mathcal{O}_X)$ is a domain for every integral scheme X. Since $X \neq \emptyset$ (by def'n of "irreducible"), $\Gamma(X, \mathcal{O}_X) \neq 0$.

Let $f, g \in \Gamma(X, \mathcal{O}_X)$ with $fg = 0$.

Write $V(f) = \{x \in X ;\ f(x) = 0 \in \kappa(x)\} \subseteq X$ (a closed subset of X; $f(x)$ denotes the image of f under the natural map $\Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x} \to \kappa(x)$ to the residue class field of X at x).

Since $fg = 0$, we have $V(f) \cup V(g) = X$, so by assumption, without loss of generality, $V(f) = X$.

Since the map $\Gamma(X, \mathcal{O}_X) \longrightarrow \prod_{x \in X} \kappa(x)$ is injective, it follows that $f = 0$.

(ii) ⇒ (i). It is clear that X is reduced.

To show that X is irreducible, it is enough to show that any two non-empty open subsets of X have non-empty intersection.

But if $U, U' \subseteq X$ are non-empty and open with $U \cap U' = \emptyset$, then $\Gamma(U \cup U', \mathcal{O}_X) \cong \Gamma(U, \mathcal{O}_X) \times \Gamma(U', \mathcal{O}_X)$ is not a domain.

Since every localization $\neq 0$ of a domain is itself a domain, the local rings of an integral scheme are domains.

Remark There exist reduced, non-irreducible schemes X s.t. $\Gamma(X, \mathcal{O}_X)$ is a field (e.g. $X = V_+(XY) \subset \mathbb{P}^2_k$, k a field).

Let X be an integral scheme. Then X has a (unique) generic point, say η, and $\mathcal{O}_{X,\eta}$ is a domain which has a unique prime ideal. In other words, $(0) \subset \mathcal{O}_{X,\eta}$ is a maximal ideal and $K(X) := \mathcal{O}_{X,\eta}$ is a field, called the <u>field of rational functions</u> or just the <u>function field</u> of X.

Similarly as in the affine case, one shows:

<u>Proposition</u> Let X be an integral scheme.

(1) For $\emptyset \neq U \subseteq X$ affine open, $K(X) = K(U) = \mathrm{Frac}(\Gamma(U, \mathcal{O}_X))$ (since the generic point of X lies in U), and for every $x \in X$, $\mathcal{O}_{X,x} \subseteq K(X)$ and $\mathrm{Frac}(\mathcal{O}_{X,x}) = K(X)$.

(2) For open subsets $\emptyset \neq V \subseteq U \subseteq X$, the restriction map and the natural map to the stalk induce injections $\Gamma(U, \mathcal{O}_X) \hookrightarrow \Gamma(V, \mathcal{O}_X) \hookrightarrow K(X)$.

(3) For $\emptyset \neq U \subseteq X$ open and an open cover $U = \bigcup_{i \in I} U_i$ by non-empty open subsets,

$$\Gamma(U, \mathcal{O}_X) = \bigcap_{i \in I} \Gamma(U_i, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_{X,x}$$

(inside $K(X)$).

Subschemes and immersions

Let X be a scheme.

A sheaf $\mathcal{J}$ is called an <u>ideal sheaf of $\mathcal{O}_X$</u> if for every $U \subseteq X$ open, $\mathcal{J}(U) \subseteq \mathcal{O}_X(U)$ is an ideal.

In this case, the sheaf (of rings) associated with the presheaf $U \mapsto \mathcal{O}_X(U)/\mathcal{J}(U)$ is called the quotient sheaf and is denoted by $\mathcal{O}_X/\mathcal{J}$.

Def. (1) Let X be a scheme. A <u>closed subscheme</u> of X is given by a closed subset $Z \subset X$ and an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ such that

$$Z = \{x \in X ; \ (\mathcal{O}_X/\mathcal{J})_x \neq 0\}$$

and such that $(Z, (\mathcal{O}_X/\mathcal{J})|_Z)$ is a scheme.

(2) A morphism $i: Z \to X$ of schemes is called a <u>closed immersion</u> if the continuous map i is a homeomorphism onto a closed subset of X and $i^b: \mathcal{O}_X \to i_* \mathcal{O}_Z$ is a <u>surjective</u> sheaf morphism.

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Remark If $Z \subseteq X$ is a closed subscheme, the inclusion $Z \hookrightarrow X$ gives rise to a closed immersion $Z \hookrightarrow X$ of schemes

If $Z \xhookrightarrow{i} X$ is a closed immersion, then $(i(Z), i_* \mathcal{O}_Z)$ is a closed subscheme of X.

Examples (1) Let $X = \operatorname{Spec} A$ be an affine scheme. For every ideal $\mathfrak{a}$, $V(\mathfrak{a}) := \operatorname{Spec} A/\mathfrak{a}$ is a closed subscheme of X.

(2) Let R be a ring and let $I \subseteq R[X_0, \dots, X_n]$ be a homogeneous ideal. Then $V_+(I)$ is a closed subscheme of $\mathbb{P}^n_R$.

(In both cases one can show that every closed subscheme has this form.)

<u>Theorem</u> Let A be a ring. The map $\mathfrak{a} \mapsto V(\mathfrak{a})$ is a bijection between the set of ideals $\subseteq A$ and the set of closed subschemes $Z \subseteq \operatorname{Spec} A$ (with inverse $Z \mapsto I_Z := \operatorname{Ker}\left(A = \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to \Gamma(Z, \mathcal{O}_Z)\right)$).

<u>Proof</u> It is easy to check that for every ideal $\mathfrak{a}$, $I_{V(\mathfrak{a})} = \mathfrak{a}$. Therefore it is enough to show that for $Z \subseteq \operatorname{Spec} A$ a closed subscheme, $Z = V(I_Z)$.

By definition of I_Z, the natural homomorphism $A \xrightarrow{\varphi} \Gamma(Z, \mathcal{O}_Z)$ factors through A/I_Z.

$$\rightsquigarrow \quad \begin{tikzcd} Z \arrow[r] \arrow[d] & \operatorname{Spec} A \\ \operatorname{Spec} A/I_Z \arrow[ur] & \end{tikzcd}$$

$\to$ Replacing A by A/I_Z we may assume that φ is injective (and then want $Z \cong \operatorname{Spec} A$).

(I) The continuous map $Z \to \operatorname{Spec} A$ is a homeomorphism.

Since this map is injective and closed, it is enough to show that it is surjective.

Let $s \in A$ with $Z \subseteq V(s)$

<u>Claim</u> There exists $N \geq 0$ s.t. $\varphi(s^N) = 0$.

<u>Proof of claim</u> Let $Z = \bigcup V_i$ be a finite affine open cover of the scheme Z.

$\rightsquigarrow \Gamma(Z, \mathcal{O}_Z) \hookrightarrow \prod_i \Gamma(V_i, \mathcal{O}_Z)$ inj.

$$\begin{tikzcd} V_i \arrow[d] & \Gamma(V_i, \mathcal{O}_Z) \\ Z \arrow[d] & \\ \operatorname{Spec} A_s & A_s \arrow[uu, "\varphi_i"'] \end{tikzcd}$$

$\rightsquigarrow$ enough to show: $\underbrace{\varphi(s^N)|_{V_i}}_{= \varphi_i(s)^N} = 0$ for $N \gg 0$,

But $U_i \subseteq Z \subseteq V(s)$ implies $U_i \subseteq V_{U_i}(\varphi_i(s))$, so $\varphi_i(s) \in \Gamma(U_i, \mathcal{O}_Z)$ is indeed nilpotent.

Since φ is injective, it follows that

$$Z \subseteq V(s) \Rightarrow s \in A \text{ nilp.} \Rightarrow V(s) = \operatorname{Spec} A.$$

Because $Z \subseteq \operatorname{Spec} A$ is closed, this implies $Z = \operatorname{Spec} A$ as topological spaces.

(II) $Z = \operatorname{Spec} A$ as schemes.

Let us identify Z and $X := \operatorname{Spec} A$ as topol. spaces. We want to show that the given sheaf homom. $\mathcal{O}_X \to \mathcal{O}_Z$ is an isomorphism.
By assumption it is surjective, so it remains to show injectivity. We can check this on stalks.

Let $\mathfrak{p} \in \operatorname{Spec} A$ and consider $A_\mathfrak{p} = \mathcal{O}_{X,\mathfrak{p}} \to \mathcal{O}_{Z,\mathfrak{p}}$
It is enough to show that for all $g \in A$ with $\frac{g}{1} \mapsto 0$ we have $g = 0$.
Fix g and let $Z = \bigcup_{i=0}^{n} V_i$ be a finite open cover such that

(1) All V_i are affine

(2) $\mathfrak{p} \in V_1 =: V$ and $\varphi(g)|_V = 0$

Let $s \in A$ with $D(s) \subseteq V$.

Claim Then ex. $N \geq 0$ s.t. $\varphi(s^N g) = 0$

Proof of claim By assumption $\varphi(g)|_V = 0$, hence $\varphi(sg)|_V = 0$. Now let $i > 1$. Since $D_{V_i}(\varphi(s)|_{V_i}) = D(s) \cap V_i \subseteq V \cap V_i$, we have $\varphi(g)|_{D_{V_i}(\varphi(s)|_{V_i})} = 0$, so the image of g in $\Gamma(V_i, \mathcal{O}_X)_{\varphi(s)|_{V_i}} = 0$ and hence there ex. $N_i \geq 0$ s.t. $\varphi(s)^{N_i}_{|V_i} \varphi(g)|_{V_i} = 0$.
We can set $N = \max\{1, N_2, \ldots, N_n\}$.

Given the claim, it follows that $s^N g = 0$ because φ is injective. It follows that $\frac{g}{1} = 0$ in A_s and a fortiori (since $\mathfrak{p} \in D(s)$) that $\frac{g}{1} = 0$ in $\mathcal{O}_{X,x}$.

Cor Every closed subscheme of an affine scheme is itself affine.

We can combine the notions of open and closed subschemes as follows:

Def Let X be a scheme. If Y is a closed subscheme of an open subscheme of X, then Y is called a subscheme of X.

If Y is a subscheme of X, then the topological space of Y is a locally closed subset of the topol space of X (i.e. the intersection of an open and a closed subset of X).

Similarly, a morphism that is a composition of open and closed immersions is called an immersion.

Schemes locally of finite type over a field

Let k be a field.

Def A k-scheme X is called locally of finite type (we also say that X is locally of finite type over k) if for every affine open $U \subseteq X$ the k-algebra $\Gamma(U, \mathcal{O}_X)$ is a k-algebra of finite type (i.e., it is finitely generated as a k-algebra).

Lemma Let X be a k-scheme and let $X = \bigcup_{i \in I} U_i$ be an affine open cover of X such that for every $i \in I$ the k-algebra $\Gamma(U_i, \mathcal{O}_X)$ is of finite type. Then X is a k-scheme locally of finite type.

Proof Omitted, see e.g. [GW, Def 3.30, Prop 3.31]

Def We say that a k-scheme X is of finite type if X is locally of finite type and quasi-compact.

Examples $\mathbb{A}^n_k$ and its closed subschemes (i.e. $V(\mathfrak{a})$, $\mathfrak{a} \subseteq k[T_1, \dots, T_n]$), $\mathbb{P}^n_k$ and all schemes $V_+(I)$, $I \subseteq k[X_0, \dots, X_n]$ homogeneous ideal, are k-schemes of finite type.

<u>Theorem</u> Let X be a k-scheme of finite type, and let $x \in X$. The following are equivalent:

(i) x is a closed point of X

(ii) the field extension $\kappa(x)/k$ is finite

(iii) the field extension $\kappa(x)/k$ is algebraic

In particular if k is algebraically closed, we get natural identifications

$$\{x \in X ;\ x \text{ closed}\} = \{x \in X ;\ \kappa(x) = k\} = X(k).$$

<u>Proof of theorem.</u> A point $x \in X$ is closed if and only if it is closed in every affine open subset of X, since X can be covered by such opens. Therefore we may assume that $X = \operatorname{Spec} A$ for a k-algebra A of finite type.

If $x \in \operatorname{Spec} A$ is closed then x corresponds to a maximal ideal $\mathfrak{m} \subset A$ and the quotient $A/\mathfrak{m}$ is a finite extension field of k by Hilbert's Nullstellensatz.

Since every finite field extension is algebraic, it only remains to show that (iii) $\Rightarrow$ (i).
So suppose that $\kappa(x)/k$ is algebraic.
We have ring homomorphisms

$$k \to A \to A/\mathfrak{p}_x \hookrightarrow \operatorname{Frac}(A/\mathfrak{p}_x) = \kappa(x)$$

(where $\mathfrak{p}_x$ denotes the prime ideal of A corresponding to x). By assumption $\kappa(x)$ is integral over k, hence the ring homomorphism $k \hookrightarrow A/\mathfrak{p}_x$ also is integral. Since k is a field it follows that $A/\mathfrak{p}_x$ is a field, so x is closed in $\operatorname{Spec} A$.

Def Let X be a topological space.

We say that a subspace $Y \subseteq X$ is <u>very dense</u> in X, if the following equivalent conditions are satisfied:

(i) The map $U \mapsto U \cap Y$ is a bijection between the set of all open subsets of X and the set of all open subsets of Y.

(ii) The map $F \mapsto F \cap Y$ is a bijection between the set of all closed subsets of X and the set of all closed subsets of Y.

(iii) For every closed subset $F \subseteq X$, we have $F = \overline{F \cap Y}$.

(iv) For all non-empty locally closed subsets $Z \subseteq X$, we have $Z \cap Y \neq \emptyset$.

Proof of equivalence: Omitted

Remark. (1) Every very dense subset of a topological space is dense (but not conversely).

(2) If $Y \subseteq X$ is very dense, we obtain an equivalence of categories between the category of sheaves on X and the category of sheaves on Y.

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Proposition Let X be a scheme locally of finite type over a field k.
Then the set X_{cl} of closed points of X is very dense in the topological space X.

Proof. We prove property (iv) in the definition of "very dense", so let $\emptyset \neq Z \subseteq X$ be locally closed.

Then there exists an open subset $U \subseteq X$ s.t. $Z \subseteq U$ is closed. Shrinking Z if necessary we may assume that Z is closed in an affine open $U \subseteq X$, say $U = \operatorname{Spec} A$, $Z = V(\mathfrak{a})$ for $\mathfrak{a} \subseteq A$. Then the topological space Z is the underlying topological space of a subscheme of X that is isomorphic to $\operatorname{Spec} A/\mathfrak{a}$.

Since A and hence also $A/\mathfrak{a}$ is a non-zero k-algebra of finite type, for every maximal ideal $\mathfrak{m} \subset A/\mathfrak{a}$ the residue class field $\kappa(\mathfrak{m}) = A/\mathfrak{m}$ is a finite extension field of k by the theorem.

Considered as a point of X the residue class field is the same field $\kappa(\mathfrak{m})$, so using the theorem again we conclude that this point is a point of $Z \cap X_{cl}$.

Remark In the above situation, using that the topol space X is sober, i.e. every irred. closed subset has a unique generic pt, one can reconstruct X as the "sobrification" of X_{cl}. On the level of sets, there are bijections

$$X \xleftrightarrow{1:1} \{Z \subseteq X \text{ closed irred.}\} \xleftrightarrow{1:1} \{Z' \subseteq X_{cl} \text{ closed irred.}\}$$

$$x \longmapsto \overline{\{x\}} \qquad\qquad Z \longmapsto Z \cap X_{cl}$$

Morphisms of schemes locally of finite type / k

Prop Let k be an algebraically closed field, and let X, Y be k-schemes of finite type. Assume that X is reduced. Let $f, g : X \to Y$ be morphisms of k-schemes.

If $f|_{X_{cl}} = g|_{X_{cl}}$ (as continuous maps) then $f = g$ (as morphisms of k-schemes).

Proof By the above theorem, for $x \in X$ and $U \subseteq X$ open with $x \in U$,

x closed in $X \iff \kappa(x) = k \iff x$ closed in U.

Therefore we may (replacing Y by an affine open $V \subseteq Y$ and X by an affine open $U \subseteq f^{-1}(V)$ and applying the argument below to all such V, U) assume that X and Y are affine.

$f^{-1}(V) = g^{-1}(V)$ since X_{cl} very dense, Say $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, where A, B are reduced k-algebras of finite type and f, g correspond to ring homomorphisms $\varphi, \psi : A \to B$.

We know that $\varphi^{-1}(\mathfrak{m}) = \psi^{-1}(\mathfrak{m})$ for every maximal ideal $\mathfrak{m} \subset B$. We want to show that $\varphi = \psi$.

Since all residue class fields at max'l ideals are $= k$ (because k is alg. closed), both φ and ψ induce the same map $\kappa(\varphi^{-1}(\mathfrak{m})) = \kappa(\psi^{-1}(\mathfrak{m})) \to \kappa(\mathfrak{m})$ for each $\mathfrak{m} \subset B$ maximal (namely, after identifying both with k, the identity).

We obtain the diagram $\to$ which is commutative for either φ or ψ, so we get $\varphi = \psi$.

$$\begin{tikzcd}
A \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] \arrow[d] & B \arrow[d, hook] \\
\prod\limits_{\substack{\mathfrak{m} \subset B \\ \text{max'l}}} \kappa(\varphi^{-1}(\mathfrak{m})) \arrow[r] & \prod\limits_{\substack{\mathfrak{m} \subset B \\ \text{max'l}}} \kappa(\mathfrak{m})
\end{tikzcd}$$

(The right vertical arrow is injective because B, being a fin. gen. k-algebra, is Jacobson (and by assumption is reduced).)

In other words, in the situation of the proposition, the natural map

$$\mathrm{Hom}_{Sch/k}(X,Y) \longrightarrow \mathrm{Map}(X(k), Y(k))$$

is injective.

(Here $\mathrm{Map}(-,-)$ denotes the set of all maps of sets, and we use this to emphasize that there is no continuity or other condition. Of course the map above is almost never surjective.)

Applying this to $U \subseteq X$ open instead of X and $Y = \mathbb{A}^1_k$, we obtain injections

$$\begin{array}{c} \Gamma(U,\mathcal{O}_X) = \mathrm{Hom}_k(k[T], \Gamma(U,\mathcal{O}_X)) = \mathrm{Hom}_{Sch/k}(U, \mathbb{A}^1_k) \\ \downarrow \\ \mathrm{Map}(U(k), k) \end{array}$$

With these results it is easy to construct, for any "reasonable" category (Var/k) of varieties over an algebraically closed field k, (i.e. "classical algebraic geometry") a fully faithful functor $(\mathrm{Var}/k) \longrightarrow$ (reduced lft k-schemes).

(For examples of such categories of varieties, see e.g. [Mumford, Red book of varieties and schemes, Ch I]
[Gathmann, Lecture notes on algebraic geometry]
[Hartshorne, Algebraic geometry, Chapter I]
[GW, Ch 1].)