V. Projective space

V.1 Gluing of schemes

Goal: describe how to "glue" a family $(U_i)_i$ of schemes, i.e. construct a scheme X so that every U_i "is" an open subscheme of X, and such that any two of these open subschemes intersect in a determined way.

Def. A gluing datum of schemes consists of the following data:

- a set I,
- a family $(U_i)_{i\in I}$ of schemes
- for all $i,j \in I$, an open subscheme $U_{ij} \subseteq U_i$ such that $U_{ii} = U_i$ for all i,
- for all $i,j \in I$, an isomorphism $\varphi_{ij} : U_{ji} \xrightarrow{\sim} U_{ij}$ such that for all $i,j,k \in I$,

"cocycle condition" $\quad \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ on $U_{ki} \cap U_{kj}$

(in particular we require that $\varphi_{jk}(U_{ki} \cap U_{kj}) \subseteq U_{ji}$).

(it follows that for all i, $\varphi_{ii} = \mathrm{id}_{U_i}$,
for all i,j, $(\varphi_{ij})^{-1} = \varphi_{ji}$,
and φ_{ij} induces an isomorphism
$$U_{ji} \cap U_{jk} \cong U_{ij} \cap U_{ik}).$$

<u>Prop</u> (Gluing of schemes) Given a gluing datum $((U_i)_{i\in I}, (U_{ij})_{i,j\in I}, (\varphi_{ij})_{i,j\in I})$, there exists a scheme X together with open immersions $\psi_i : U_i \to X$ s.t. for all $i,j \in I$, $\psi_j \circ \varphi_{ji} = \psi_i$ on U_{ij}, $X = \bigcup_{i\in I} \psi_i(U_i)$, and
$$\psi_i(U_{ij}) = \psi_i(U_i) \cap \psi_j(U_j) = \psi_j(U_{ji}).$$
The scheme X together with the ψ_i is uniquely determined up to unique isomorphism.

21.12.2022

By "gluing of morphisms", X together with the ψ_i satisfies the following universal property:

For every scheme T and every family $\xi_i : U_i \to T$ of morphisms such that $\xi_j \circ \varphi_{ji} = \xi_i$ for all i,j, there exists a unique morphism $\xi : X \to T$ with $\xi \circ \psi_i = \xi_i \; \forall i \in I$.

In particular, this implies the uniqueness statement of the proposition.

<u>Proof</u> (of existence).

As a set, we define X as the set $\coprod_{i \in I} U_i / \sim$ of equivalence classes for the following equivalence relation on the disjoint union $\coprod U_i$:

For $x_i \in U_i$, $x_j \in U_j$, $x_i \sim x_j \iff x_i \in U_{ij}, x_j \in U_{ji}$, and $\varphi_{ji}(x_i) = x_j$.

The "cocycle condition" $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ implies that this is an equivalence relation.

Denote by $\varphi_i : U_i \to \coprod U_i \twoheadrightarrow X$ the natural map (of sets). (Note that the φ_i are injective.)

We equip X with the finest topology such that all the maps φ_i are continuous. More concretely, a subset $U \subseteq X$ is open if and only if for all $i \in I$, $\varphi_i^{-1}(U) \subseteq U_i$ is open.

In particular, $X = \bigcup_{i \in I} \varphi_i(U_i)$ is an open cover of X, and for all i, j, we have homeomorphisms

$$U_{ij} \xrightarrow{\sim} \varphi_i(U_{ij}) = \varphi_i(U_i) \cap \varphi_j(U_j) = \varphi_j(U_{ji}) \xleftarrow{\sim} U_{ji}.$$

To define a sheaf $\mathcal{O}_X$ of rings on X, recall that it is enough to define it on a basis of the topology of X. It is therefore enough to define $\mathcal{O}_X(U)$ (and restriction maps) for all $U \subseteq X$ open such that there exists $i \in I$ with $U \subseteq \varphi_i(U_i)$. For each U we fix such an index i and define $\mathcal{O}_X(U) := \mathcal{O}_{U_i}(\varphi_i^{-1}(U))$.

Whenever $U \subseteq \varphi_i(U_i) \cap \varphi_j(U_j)$ for $i, j \in I$, we can identify $\mathcal{O}_{U_i}(\varphi_i^{-1}(U))$ with $\mathcal{O}_{U_j}(\varphi_j^{-1}(U))$ using the isom. φ_{ij}, φ_{ji}. Making these identifications, $\mathcal{O}_X(U)$ is independent of the choices and in particular we obtain well-defined restriction maps.

Altogether we have defined a ringed space $(X, \mathcal{O}_X)$ s.t. for every i, $(U_i, \mathcal{O}_{U_i}) \cong (\varphi_i(U_i), \mathcal{O}_X|_{\varphi_i(U_i)})$.

In particular, X is a scheme. By construction, it has all the properties stated in the proposition.

Example (Disjoint union) Given any family $(U_i)_{i \in I}$ of schemes, we can define a gluing datum by setting $U_{ij} := \emptyset$ for all i, j. The resulting scheme X obtained by gluing is called the disjoint union of the schemes U_i and denoted by $\coprod_{i \in I} U_i$.

Example. In the case where $I = \{1,2\}$ has only two elements, a gluing datum corresponds to schemes U_1, U_2, open subschemes $U_{12} \subseteq U_1$, $U_{21} \subseteq U_2$ and an isomorphism $\varphi : U_{12} \xrightarrow{\sim} U_{21}$.

Let X be the scheme obtained by gluing. We view U_1, U_2 as open subschemes of X. For $V \subseteq X$ open,

$$\Gamma(V, \mathcal{O}_X) = \{ (s_1, s_2) \in \Gamma(V \cap U_1, \mathcal{O}_{U_1}) \times \Gamma(V \cap U_2, \mathcal{O}_{U_2}) \, ; \; s_1|_{V \cap U_{12}} = \varphi^\flat (s_2|_{V \cap U_{21}}) \}.$$

Example. k a field, $U_1 = U_2 = \mathbb{A}^1_k$,

$U_{12} = U_{21} = \mathbb{A}^1_k \setminus \{0\}$,

$\varphi : U_{12} \xrightarrow{\mathrm{id}} U_{21}$

$\overset{\text{gluing}}{\rightsquigarrow}$ X "affine line with doubled origin"

The scheme X is not affine.

V.2 Projective space

In the introduction we defined projective space $\mathbb{P}^n(k)$ over a field k, $\mathbb{P}^n(k) = (k^{n+1} \setminus \{0\})/k^\times$.

Goal: Construct a k-scheme $\mathbb{P}^n_k$ s.t.

$\mathbb{P}^n_k(k)$ "is" the projective space $/k$ as above.

We can just as well carry out the basic construction over any ring R, so we will do that. On the other hand, it is (possible, but) not so easy to develop methods that would allow us to construct the scheme $\mathbb{P}^n_k$ as a quotient of $\mathbb{A}^{n+1}_k \setminus \{0\}$ (or even to define what the word "quotient" should mean in this context, so we will start out from the following description:

for $i \in \{0, \dots, n\}$, let $U_i = \{(x_0 : \dots : x_n) \in \mathbb{P}^n(k);\ x_i \neq 0\}$,

then $U_i \subseteq \mathbb{P}^n(k)$ open (w.r.t. the Zariski topology), omitted

$\mathbb{P}^n(k) = \bigcup_i U_i$ and $U_i \xrightarrow{1:1} k^n$, $(x_0 : \dots : x_n) \mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right)$

Motivated by this description, we will construct $\mathbb{P}^n_k$ by gluing $n+1$ copies of $\mathbb{A}^n_k$ in the "same way" as the U_i could be used to obtain $\mathbb{P}^n(k)$ by gluing.

So we now change notation and define, for a fixed base ring R,

$$U_i := \mathbb{A}^n_R = \operatorname{Spec} \underbrace{R\left[\frac{X_0}{X_i}, \dots, \widehat{\frac{X_i}{X_i}}, \dots, \frac{X_n}{X_i}\right]}$$

$i = 0, \dots, n$,

and for $i, j \in \{0, \dots, n\}$, let

$U_{ij} = D\left(\frac{X_j}{X_i}\right) \quad (\subseteq U_i)$

$= \operatorname{Spec} R\left[\frac{X_0}{X_i}, \dots, \widehat{\frac{X_i}{X_i}}, \dots, \frac{X_n}{X_i}, \frac{X_i}{X_j}\right]$

Then $U_{ij} = U_{ji}$ and these identifications define a gluing datum.

- we view all these rings (for varying i) as subrings of $R[X_0, \dots, X_n, X_0^{-1}, \dots, X_n^{-1}]$ $= R[X_0, \dots, X_n]_{X_0 \cdots X_n}$
- clearly $R\left[\frac{X_0}{X_i}, \dots, \widehat{\frac{X_i}{X_i}}, \dots, \frac{X_n}{X_i}\right]$ is isomorphic to a polynomial ring $/R$ in n variables; the "strange choice" of variables will allow us to write down the gluing isomorphisms very easily

(Since all the isom. $U_{ij} = U_{ji}$ "are" the identity, the cocycle condition holds for trivial reasons.)

By the proposition on gluing of schemes, this defines a scheme $\mathbb{P}^n_R$ which by definition admits an open cover $\mathbb{P}^n_R = \bigcup_{i=0}^n U_i$

(where we identify each U_i with its image in $\mathbb{P}^n_R$) such that $U_i \cap U_j = U_{ij} (= U_{ji})$.

Since all the U_i are R-schemes and the identifications $U_{ij} = U_{ji}$ are isomorphisms of R-schemes, $\mathbb{P}^n_R$ is an R-scheme in a natural way.

<u>Proposition</u> Let R be a ring, $n \geq 0$. The natural map $R \to \Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R})$ (coming from the structure morphism $\mathbb{P}^n_R \to \operatorname{Spec} R$) is a ring isomorphism.

<u>Cor.</u> If $n \geq 1$ and $R \neq 0$, then the scheme $\mathbb{P}^n_R$ is not affine.

10.1.2023

Proof of corollary: If $\mathbb{P}^n_R$ is affine, then by the proposition, the structure morphism $\mathbb{P}^n_R \to \operatorname{Spec} R$ is an isomorphism, and in particular injective on topological spaces. But then $\mathbb{A}^n_R \hookrightarrow \mathbb{P}^n_R \to \operatorname{Spec} R$ is also injective and thus $n=0$.

Proof of proposition. We use the cover $\mathbb{P}^n_R = \bigcup U_i$ and the sheaf property of $\mathcal{O}_{\mathbb{P}^n_R}$ to compute the global sections.

We can view $\Gamma(U_i, \mathcal{O}_{\mathbb{P}^n_R})$ and $\Gamma(U_i \cap U_j, \mathcal{O}_{\mathbb{P}^n_R})$ as subrings of $R[X_0, \dots, X_n, X_0^{-1}, \dots, X_n^{-1}]$, and the restriction map is just the inclusion map. So we obtain an identification

$$\Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}) = \bigcap_{i=0}^{n} R\left[\frac{X_0}{X_i}, \dots, \widehat{\frac{X_i}{X_i}}, \dots, \frac{X_n}{X_i}\right] = R.$$

V.3 Zero sets of homogeneous ideals

Recall (The "classical setting as in the Introduction)

Let k a field, $f_1, \dots, f_r \in k[X_0, \dots, X_n]$ homogeneous polynomials

$\rightsquigarrow V_+(f_1, \dots, f_r) \subseteq \mathbb{P}^n(k)$ set of common zeros.

For $U_i = \{(x_0 : \dots : x_n) ; x_i \neq 0\}$, we obtain (with $\Phi_i(f) = f(\frac{X_0}{X_i}, \dots, 1, \dots, \frac{X_n}{X_i})$),

$$\left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right) \longleftarrow\!\shortmid (x_0 : \dots : x_n)$$

$$\begin{array}{ccccc} \mathbb{A}^n(k) & = & U_i & \hookrightarrow & \mathbb{P}^n(k) \\ \cup\!| & & \cup\!| & & \cup\!| \\ V(\Phi_i(f_1), \dots, \Phi_i(f_r)) & = & V_+(f_1, \dots, f_r) \cap U_i & \hookrightarrow & V_+(f_1, \dots, f_r). \end{array}$$

To carry out an analogous construction for schemes (i.e., a scheme $V_+(f_1, \dots, f_r)$), let R be a ring and let $I \subseteq R[X_0, \dots, X_n]$ a homogeneous ideal

(i.e., I admits a generating system consisting of homogeneous elements).

Define $V_i := V(\Phi_i(f),\ f \in I \text{ homogeneous}) \subseteq U_i$,

$$V_{ij} := V_i \cap U_{ij} \quad (\subseteq V_i \subseteq U_i)$$

The identification $U_{ij} = U_{ji}$ restricts to an identification $V_{ij} = V_{ji}$ because

$$X_j^d\, \Phi_i(f) = X_i^d\, \Phi_j(f) \text{ for } f \text{ homog. of degree } d,$$

therefore the above ideals coincide if $\frac{X_i}{X_j}$ is a unit.

We obtain a gluing datum, and by gluing obtain a scheme $V_+(I)$ together with an open cover $V_+(I) = \bigcup V_i$ and a morphism $V_+(I) \xrightarrow{\iota} \mathbb{P}^n_R$ which restricted to V_i is the inclusion $V_i \hookrightarrow U_i \hookrightarrow \mathbb{P}^n_R$.

In particular, $V_+(I) \to \mathbb{P}^n_R$ is a homeomorphism onto a closed subset of $\mathbb{P}^n_R$. Furthermore, the sheaf morphism $\mathcal{O}_{\mathbb{P}^n_R} \longrightarrow \iota_* \mathcal{O}_{V_+(I)}$ is surjective. (Both claims can be checked on the open cover $\mathbb{P}^n_R = \bigcup U_i$.)

This construction "recovers" the classical case recalled above in the sense that for k a field, $f_1, \dots, f_r \in k[X_0, \dots, X_n]$ homogeneous,

$$V_+(f_1, \dots, f_r)(k) = \{(x_0 : \dots : x_n) ; \forall j : f_j(x_0, \dots, x_n) = 0\}$$

(when we write elements of $\mathbb{P}^n_k(k)$ using homogeneous coordinates).

<u>Notation</u> R a ring, $f \in R[X_0, \dots, X_n]$ homog.

$\leadsto$ $D_+(f) = \mathbb{P}^n_R \setminus V_+(f)$ $\quad (\subseteq \mathbb{P}^n_R$ open$)$

Special case: $D_+(X_i) = U_i$ (with notation as above).