VII. Fiber products

VII.1 Fiber products in arbitrary categories

Let $\mathcal{C}$ be a category.

Def Let $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ be morphisms in $\mathcal{C}$. An object Z of $\mathcal{C}$ together with morphisms $Z \xrightarrow{p} X$, $Z \xrightarrow{q} Y$ is called a <u>fiber product</u> of f and g (or: of X and Y over S) if $f \circ p = g \circ q$ and the following universal property is satisfied:

For every object T of $\mathcal{C}$ together with morphisms $u: T \to X$, $v: T \to Y$ s.t. $u \circ p = v \circ q$, then exists a unique morphism $w: T \to Z$ s.t.

$p \circ w = u$, $q \circ w = v$.

$$
\begin{tikzcd}
T \arrow[dr, dashed, "\exists! w"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& Z \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & S
\end{tikzcd}
$$

If a fiber product (of X, Y over S) exists, then it is unique up to unique isomorphism and is denoted by $X \times_S Y$. The morphisms $X \times_S Y \to X$, $X \times_S Y \to Y$ are called the projections.

Example Let $\mathcal{C} = (\text{Sets})$. Then for every pair $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ of maps of sets,

$$X \times_S Y := \{ (x, y) \in X \times Y \ ; \ f(x) = g(y) \}$$
$$= \bigcup_{s \in S} f^{-1}(s) \times g^{-1}(s) \qquad \subseteq X \times Y$$

hence the name "fiber product"

together with the restrictions of the projections $X \times Y \to X$, $X \times Y \to Y$ to $X \times_S Y$ is a fiber product of X and Y over S. So in (Sets) all fiber products exist.

Important special cases:

- $S = \{*\}$ a singleton, X, Y sets, $X \to S$, $Y \to S$ the unique maps.

  Then $X \times_S Y = X \times Y$ is the product of X and Y.

  (More generally if $\mathcal{C}$ is any category which has a terminal object S then the universal property of fiber products over S coincides with the universal property of the product.)

- $f : X \to S$ a map of sets, $s \in S$, $Y = \{s\}$, $Y \to S$, $s \mapsto s$. Then the natural map $f^{-1}(s) \to X \times_S \{s\}$ (given by the universal property of the fiber product, the inclusion $f^{-1}(s) \hookrightarrow X$ and the constant map $f^{-1}(s) \to \{s\}$) is a bijection, i.e. an isomorphism in $\mathcal{C}$.

(The analogous statement is true also in the category of topological spaces.)

Remark. Let $\mathcal{C}$ be a category, $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ morphisms s.t. the fiber product $X \times_S Y$ exists.

Then we can express the universal property of $X \times_S Y$ as follows: For every $T \in \mathrm{Ob}\,\mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}}(T, X \times_S Y) = \mathrm{Hom}_{\mathcal{C}}(T, X) \times_{\mathrm{Hom}_{\mathcal{C}}(T,S)} \mathrm{Hom}(T, Y)$$

(where "=" means that the natural map $\rightarrow$ given by composition with the projections is a bijection).

VII.2 Fiber products of schemes

<u>Theorem</u> Let $\mathcal{C} = (\mathrm{Sch})$ be the category of schemes. For all morphisms

$f: X \to S, \quad g: Y \to S$ of schemes

the fiber product $X \times_S Y$ exists.

If $X = \mathrm{Spec}\, A$, $Y = \mathrm{Spec}\, B$, $S = \mathrm{Spec}\, R$ are affine, then $X \times_S Y = \mathrm{Spec}\, A \otimes_R B$.

The theorem will allow us to define "good" notions of products of schemes and of fibers of scheme morphisms.

Proof. Since the tensor product of rings has the universal property "opposite" to the universal property of fiber products, it is immediate that $\operatorname{Spec} A \otimes_R B$ together with the natural maps to $\operatorname{Spec} A$ and $\operatorname{Spec} B$ is a fiber product of $\operatorname{Spec} A$ and $\operatorname{Spec} B$ over $\operatorname{Spec} R$ in the category of affine schemes (given ring homom. $R \to A$, $R \to B$ or equivalently scheme morphisms $\operatorname{Spec} A \to \operatorname{Spec} R$, $\operatorname{Spec} B \to \operatorname{Spec} R$).

But in fact we can say more:

Given any scheme T, we have

$$\underbrace{\operatorname{Hom}_{Sch}(T, \operatorname{Spec}(A \otimes_R B)) = \operatorname{Hom}_{Ring}(A \otimes_R B, \Gamma(T, \mathcal{O}_T))}_{\text{"morphisms into affine schemes"}}$$

$$\underbrace{= \operatorname{Hom}_{Ring}(A, \Gamma(T,\mathcal{O}_T)) \times_{\operatorname{Hom}(R, \Gamma(T,\mathcal{O}_T))} \operatorname{Hom}(B, \Gamma(T,\mathcal{O}_T))}_{\text{univ. prop. of } \otimes \text{ of } R\text{-algebras}}$$

$$= \operatorname{Hom}_{Sch}(T, \operatorname{Spec} A) \times_{\operatorname{Hom}_{Sch}(T, \operatorname{Spec} R)} \operatorname{Hom}_{Sch}(T, \operatorname{Spec} B)$$

This proves the final assertion.

The general case follows by covering

$$S = \bigcup W_i, \quad f^{-1}(W_i) = \bigcup U_{ij}, \quad g^{-1}(W_i) = \bigcup V_{ik}$$

with affine open W_i, U_{ij}, V_{ik} and constructing a gluing datum for the family $U_{ij} \times_{W_i} V_{ik}$.

The key point here is that the isomorphisms between the desired "intersections" arise by the uniqueness part of the universal property of the fiber product and hence themselves satisfy a uniqueness property so that the cocycle condition must hold.

For further details see e.g. [GW] Thm 4.18.

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Notation If $X \to S$, $Y \to S$ are scheme morphisms with $Y = \operatorname{Spec} B$ affine, then we also write $X \otimes_S B := X \times_S Y$.

If also $S = \operatorname{Spec} R$ is affine, we also write $X \otimes_R B = X \times_S Y$.

If $S = \operatorname{Spec} R$ is affine (but neither X nor Y necessarily is affine) we sometimes write $X \times_R Y$ for $X \times_S Y$.

Def. (Product of schemes) Let X, Y be schemes. Then $X \times Y := X \times_{\operatorname{Spec} \mathbb{Z}} Y$ (fiber product of the unique morphisms $X \to \operatorname{Spec} \mathbb{Z}$, $Y \to \operatorname{Spec} \mathbb{Z}$) is called the product of X and Y.

Remark Let S be a scheme.

Morphisms $f: X \to S$, $g: Y \to S$ can be seen as objects of the category (Sch/S) of S-schemes and the fiber product $X \times_S Y$ of f and g naturally is a S-scheme. In this sense:

fiber products over S "=" products in (Sch/S).

Remark As pointed out above, fiber product behaves well w.r.t. T-valued points: $(X \times_S Y)(T) = X(T) \times_{S(T)} Y(T)$. If S is a scheme, X, Y and T are S-schemes and we consider T-valued pts in the sense of S-schemes, then $(X \times_S Y)(T) = X(T) \times Y(T)$.

Examples (1) Let R be a ring, $n, m \geq 0$.

Then $\mathbb{A}^n_R \times_R \mathbb{A}^m_R \cong \mathbb{A}^{n+m}_R$.

(2) Let k be a field. If X, Y are k-schemes lft, then $X \times_k Y$ is lft, and $(X \times_k Y)(k) = X(k) \times Y(k)$ (cf. the previous remark).

If k is algebraically closed, then we obtain $(X \times_k Y)_{cl} = X_{cl} \times Y_{cl}$ as sets (but usually not as topological spaces).

(3)
$$\begin{aligned} \operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C} &= \operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \\ &\cong \operatorname{Spec}(\mathbb{C} \times \mathbb{C}) \\ &\cong \operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C} \end{aligned}$$

$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ ring isomorphism

has two (closed) points (!)

(Note: if $n, m \geq 1$, $R \neq 0$, then $\mathbb{P}^n_R \times \mathbb{P}^m_R \not\cong \mathbb{P}^{n+m}_R$.)

VII.3 Fibers of morphisms of schemes

<u>Def.</u> Let $f: X \to S$ be a morphism of schemes. Let $s \in S$ and let $g = i_s : \mathrm{Spec}\, \kappa(s) \to S$ be the natural morphism. The fiber product $X \times_S \mathrm{Spec}\, \kappa(s)$ (often denoted by $X \otimes_S \kappa(s)$ or just by $f^{-1}(s)$ or X_s) is called the <u>(scheme-theoretic) fiber of f over s</u>

By the following proposition, we can view the definition as providing a scheme structure on the "topological" fibers.

<u>Proposition</u> In the situation of the proposition the underlying continuous map of the projection $X \otimes_S \kappa(s) \to X$ is a homeomorphism between the topological space of $X \otimes_S \kappa(s)$ and the fiber $f^{-1}(s)$ of the continuous map $f: X \to S$ (with the subspace topology for the inclusion $f^{-1}(s) \subseteq X$).

Proof. Replacing S by an affine open neighborhood of s and then considering an affine open cover of X, we reduce to the case that X and S are affine.

In this case, the result follows easily from the descriptions of the spectrum of a localization / a quotient of a ring in terms of the spectrum of the original ring:

Say $X = \operatorname{Spec} A$, $S = \operatorname{Spec} R$, $s \longleftrightarrow \mathfrak{p} \subset R$.

$$X \otimes_S \kappa(s) = \operatorname{Spec} A \otimes_R \kappa(\mathfrak{p}) = \operatorname{Spec} A \otimes R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$$

$$= \operatorname{Spec}(S^{-1}A/\mathfrak{P}) \overset{1:1}{\longleftrightarrow}$$

$\varphi : R \to A$

$S = \varphi(R \setminus \mathfrak{p})$

$\mathfrak{P} \subset S^{-1}A$ the ideal generated by $\mathfrak{p}$

$$\{\mathfrak{q} \in \operatorname{Spec} A;\ \mathfrak{P} \subset \mathfrak{q},\ S \cap \mathfrak{q} = \emptyset\}$$

$$= \{\mathfrak{q} \in \operatorname{Spec} A;\ \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}\}.$$

Example (1) Let k be an alg. closed field, char $k \neq 2$.

Let $f: V(X^2+Y) \longrightarrow \mathbb{A}^1_k = \operatorname{Spec} k[Y]$

$$\begin{array}{c} V(X^2+Y) \\ \cap \\ \mathbb{A}^2_k = \operatorname{Spec} k[X,Y] \end{array}$$

be the scheme morphism corresponding to the k-algebra homom. $k[Y] \to k[X,Y]/(X^2+Y)$.

$$Y \longmapsto Y$$

For $y \in k = \mathbb{A}^1_k(k) = \mathbb{A}^1_{k,cl}$, the scheme-theor fiber $f^{-1}(y)$ is

$$\operatorname{Spec}\left(k[X,Y]/_{X^2+Y} \otimes_{k[Y]} \kappa(y) \right)$$

$$\cong \operatorname{Spec} k[X]/_{X^2+y} \cong \begin{cases} \operatorname{Spec}(k \times k) & y \neq 0 \\ \operatorname{Spec} k[X]/_{X^2} & y = 0. \end{cases}$$

In all cases the vector space $\Gamma(f^{-1}(y), \mathcal{O}_{f^{-1}(y)})$ has dimension 2 over k. The scheme $f^{-1}(y)$ is reduced if and only if $y \neq 0$.

Remark Let $f : X \to S$ be a scheme morphism. For every $s \in S$ the fiber $f^{-1}(s)$ is a $\kappa(s)$-scheme in a natural way (via the projection $X \otimes_S \kappa(s) \to \operatorname{Spec} \kappa(s)$).

Thus the morphism f gives rise to a family of schemes over fields indexed by S (which could be considered as "more classical" objects).