Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2018/19

Problem sheet 7

Due date: Nov. 27, 2018.

Problem 29

Let X be a topological space and let $f\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X.

(1) We define the *image* $\mathrm{im}(f)$ *of* f as the sheaf associated with the presheaf

$$U \mapsto \mathrm{im}(\mathscr{F}(U) \to \mathscr{G}(U)), \quad U \subseteq X \text{ open}.$$

Prove that f induces a surjective morphism $\mathscr{F} \to \mathrm{im}(f)$ of sheaves.

(2) Now assume that f above is a morphism of sheaves of abelian groups. We define the *kernel* $\ker(f)$ *of* f as the sheaf

$$U \mapsto \ker(\mathscr{F}(U) \to \mathscr{G}(U)), \quad U \subseteq X \text{ open}.$$

Prove that this is in fact a sheaf and that f induces an injective morphism $\ker(f) \to \mathscr{F}$.

Problem 30

Give an example of affine schemes X, Y and a morphism $X \to Y$ of ringed spaces which is not a morphism of locally ringed spaces.

Problem 31

Let A be a domain with field of fractions K. We view all (non-trivial) localizations of A as subrings of K. Let $f \in A$, $f \neq 0$. Show that

$$A_f = \bigcap_{f\in A, f\notin\mathfrak{p}} A_{\mathfrak{p}}.$$

Hint. Given $g \in \bigcap_{f\in A, f\notin\mathfrak{p}} A_{\mathfrak{p}}$, consider the ideal

$$\mathfrak{a} = \{h \in A;\ hg \in A\} \subseteq A.$$

Problem 32

Let k be an algebraically closed field of characteristic $\neq 2$. Let $X = D(T+1) \subseteq \mathbb{A}^1_k$ (where T is the coordinate on $\mathbb{A}^1_k$, i.e., $\mathbb{A}^1_k = \operatorname{Spec} k[T]$), and let $Y = V(U^2 - T^2(T+1)) \subseteq A^2_k$ (with coordinates T, U). We view Y as the scheme $\operatorname{Spec} k[T,U]/(U^2 - T^2(T+1))$.

Show that there is a morphism $f\colon X \to Y$ of schemes which on closed points is given as $t \mapsto (t^2-1, t(t^2-1))$.

Show that f is a bijection on the underlying topological spaces, but not an isomorphism of schemes.

Hint. You may make use of Hilbert's Nullstellensatz and of the fact that $\dim X = \dim Y = 1$, i.e., all non-zero prime ideals in the affine coordinate rings of X and Y are maximal.