Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2018/19

Problem sheet 5

Due date: Nov. 13, 2018.

Problem 17

(1) Let $\psi\colon A \to B$ be an injective ring homomorphism between reduced rings. Show that every minimal prime ideal of A is in the image of ${}^a\psi$. Give an example where ${}^a\psi$ is not surjective.

(2) Let $\varphi\colon A \to B$ be a ring homomorphism, and let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the map attached to φ. Assume that f is bijective and that f reflects specialization, i.e., for $x, x' \in \operatorname{Spec} B$ we have $f(x') \in \overline{\{f(x)\}}$ if *and only if* $x' \in \overline{\{x\}}$. Show that f is a homeomorphism.

Hint. In (2), we need to show that f is closed. Let $\mathfrak{b} \subset B$ be a radical ideal. Apply (1) to the ring homomorphism $A/\varphi^{-1}(\mathfrak{b}) \to B/\mathfrak{b}$ induced by φ and then use that f reflects specialization to show that $f(V(\mathfrak{b})) = V(\varphi^{-1}(\mathfrak{b}))$

Problem 18

Give an example of a continuous map $f\colon X \to Y$ between topological spaces X, Y which is bijective and compatible with specialization (i.e., for $x, x' \in X$ we have $x' \in \overline{\{x\}}$, if and only if $f(x') \in \overline{\{f(x)\}}$) which is not a homeomorphism.

Problem 19

Let X be a topological space and let $(U_i)_i$ be an open covering of X. For all i let $\mathscr{F}_i$ be a sheaf on U_i. Assume that for each pair (i, j) of indices we are given isomorphisms $\varphi_{ij}\colon \mathscr{F}_{j|U_i\cap U_j} \xrightarrow{\sim} \mathscr{F}_{i|U_i\cap U_j}$ satisfying for all i, j, k the "cocycle condition"

$$\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk} \quad \text{on } U_i \cap U_j \cap U_k.$$

Show that there exists a sheaf $\mathscr{F}$ on X and for all i isomorphisms $\psi_i\colon \mathscr{F}_i \xrightarrow{\sim} \mathscr{F}_{|U_i}$ such that $\psi_i \circ \varphi_{ij} = \psi_j$ on $U_i \cap U_j$ for all i, j. Show that $\mathscr{F}$ and the ψ_i are uniquely determined up to unique isomorphism by these conditions.

Remark: We say that the sheaf $\mathscr{F}$ is obtained by *gluing the $\mathscr{F}_i$* via the *gluing datum* φ_{ij}.

Problem 20

Let X be a topological space, let Z be a subset of X (equipped with the subspace topology) and let $\iota\colon Z \to X$ be the inclusion map. Let $\mathscr{F}$ be a sheaf on Z.

(1) Let $\overline{Z}$ denote the closure of Z in X, and let $x \in X \smallsetminus \overline{Z}$. Show that the stalk $(\iota_*\mathscr{F})_x$ is a singleton set.

(2) Let $x \in Z$. Show that there is a natural isomorphism $(\iota_*\mathscr{F})_x \cong \mathscr{F}_x$.