Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2018/19

Problem sheet 11

Due date: Jan. 8, 2018.

Problem 45

Given an example of a field k and an irreducible homogeneous polynomial $f \in k[X, Y, Z]$ such that $V_+(f)$ (a closed subscheme of $\mathbb{P}^2_k$) is not isomorphic to $\mathbb{P}^1_k$ as a k-scheme.

Problem 46

A topological space X is called noetherian, if it satisfies the descending chain condition for closed subsets, i.e., every descending chain

$$X \supseteq Z_1 \supseteq Z_1 \supseteq \cdots$$

of closed subsets is stationary.

(1) Show that every subspace of a noetherian topological space is noetherian.

(2) Let X be a noetherian topological space. Show that every open subset of X is quasi-compact.

(3) Let X be a noetherian topological space. Show that X has only finitely many irreducible components.

(4) Let X be a noetherian scheme. Show that the underlying topological space is noetherian.

(5) Give an example of a non-noetherian scheme X whose underlying topological space is noetherian.

Problem 47

1. Let X be an irreducible noetherian scheme with generic point η such that the local ring $\mathscr{O}_{X,\eta}$ is reduced. Show that there exists a non-empty open subscheme $U \subseteq X$ which is reduced.
2. Give an example of a non-reduced irreducible noetherian scheme X such that the local ring at the generic point is reduced.

Problem 48

Let k be an algebraically closed field.

(1) Show that there exists a unique radical ideal $I \subseteq k[T_0, T_1, T_2]$ such that for $Z := V(I) \subseteq \mathbb{A}^3_k$ we have

$$Z(k) = \{(t, t^2, t^3);\ t \in k\} \subset k^3 = \mathbb{A}^3_k(k).$$

Show that $Z \cong \mathbb{A}^1_k$.

(2) Compute the closure $\overline{Z}$ of Z in $\mathbb{P}^3_k$ (with respect to the embedding $\mathbb{A}^3_k = D_+(X_0) \subseteq \mathbb{P}^3_k$, where we use X_0, X_1, X_2, X_3 as homogeneous coordinates on $\mathbb{P}^3_k$), and give a homogeneous ideal $J \subseteq k[X_0, X_1, X_2, X_3]$ such that $V_+(J)$ has underlying topological space $\overline{Z}$.