Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2018/19

Problem sheet 10

Due date: Dec. 18, 2018.

Problem 41

Let R be a ring, $n \geqslant 1$, and let $A = (a_{ij})_{i,j} \in GL_{n+1}(R)$ be an invertible $(n+1) \times (n+1)$-matrix with entries in R. Let $A = R[X_0, \dots, X_n]$. Let $Z = V(X_0, \dots, X_n) \subseteq \mathbb{A}_R^{n+1} = \operatorname{Spec} A$, and let $U = \mathbb{A}_R^{n+1} \smallsetminus Z$, an open subscheme of $\mathbb{A}_R^{n+1}$. Recall the morphism $U \to \mathbb{P}_R^n$ of R-schemes. (In coordinates, we think of $(x_0, \dots, x_n) \mapsto (x_0 \colon \cdots : x_n)$.)

(1) The ring isomorphism

$$A \to A, \quad X_i \mapsto \sum_j a_{ij} X_j,$$

induces an isomorphism $\mathbb{A}_R^{n+1} \to \mathbb{A}_R^{n+1}$ of R-schemes.

Show that A restricts to an automorphism f_A of U.

(2) Show that there exists a unique automorphism f_A of $\mathbb{P}_R^n$ which fits into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f_A} & U \\ \downarrow & & \downarrow \\ \mathbb{P}_R^n & \xrightarrow{f_A} & \mathbb{P}_R^n. \end{array}$$

In this way we obtain a group homomorphism from $GL_{n+1}(R)$ into the group $\operatorname{Aut}_R(\mathbb{P}_R^n)$ of automorphisms of the R-scheme $\mathbb{P}_R^n$.

(3) Now let k be a field. We identify $\mathbb{P}^1(k) = D_+(X_0)(k) \cup V_+(X_0)(k) = k \cup \{\infty\}$. Let $x, y, z \in \mathbb{P}^1(k)$ be distinct points. Show that there exists a unique automorphism f of $\mathbb{P}_k^1$ such that f is of the form f_A and such that

$$f(0) = x, \quad f(1) = y, \quad f(\infty) = z.$$

Problem 42

Let k be an algebraically closed field of characteristic $\neq 2$. Let $f \in k[X_0, \dots, X_n]$ be homogeneous of degree 2, $f \neq 0$. We call $V_+(f) \subseteq \mathbb{P}_k^n$ a *quadric*. Which of the following quadrics in $\mathbb{P}_k^2$ are isomorphic as k-schemes?

$$V_+(X_0^2 + X_1^2), \quad V_+(X_0^2 + X_1^2 + X_2^2), \quad V_+(X_0 X_2 - X_1^2).$$

Show that $V_+(X_0 X_2 - X_1^2) \cong \mathbb{P}_k^1$.

Problem 43

Let X be a scheme, $x, y \in X$, $x \neq y$. Show that there exists an open subset $U \subset X$ such that U contains exactly one of x, y.

Hint. First reduce to the case of an affine scheme.

Problem 44

Let $f\colon X \to Y$ be a morphism of schemes. Let $V = \bigcup_i V_i$ be a cover by affine open subschemes such that for all i, $f^{-1}(V_i)$ is quasi-compact. Show that f is a *quasi-compact* morphism, i.e., $f^{-1}(V)$ is quasi-compact for every quasi-compact open $V \subseteq Y$.